March 2, 2011

Asymptotic Parametrices of Elliptic Edge Operators

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Abstract. We study operators on a singular manifold, here of conical or edge type, and develop a new general approach of representing asymptotics of solutions to elliptic equations close to the singularities. The idea is to construct so-called asymptotic parametrices with flat left-over terms. Our structures are motivated by models of particle physics with singular potentials that contribute embedded singularities in $\mathbb{R}^N$ of higher order, according to the number of particles.

CONTENTS

Introduction

Ellipticity of an operator A on a singular configuration M, here with a stratification $s(M) := (s_j(M))_{j=0,\dots,k}$, cf. the definition below, relies on a principal symbolic structure $\sigma(A) := (\sigma_j(A))_{j=0,\dots,k}$, more precisely, on a componentwise invertibility (or Fredholm) condition, cf. [24]. In the simplest case when M is smooth (which corresponds to $k = 0$) and if A is a classical pseudo-differential operator of order

$\mu \in \mathbb{R}$, e.g., a differential operator with smooth coefficients, then $\sigma_0(A)$ is the standard homogeneous principal symbol of order μ. The case $k = 1$ corresponds to conical or edge singularities. Those form a subset $s_1(M) \subset M$, and we set $s_0(M) := M \setminus s_1(M)$; both $s_0(M)$ and $s_1(M)$ are smooth manifolds. Now $\sigma_0(A)$ is the principal symbol of the operator A on $s_0(M)$, while $\sigma_1(A)$ is an operator-valued symbol associated with $s_1(M)$.

In the case of conical singularities, i.e. $\dim s_1(M) = 0$, without loss of generality we assume that $s_1(M)$ consists of a single point v. Then M is locally near v modelled on a cone

$$(0.1) \qquad X^{\Delta} := (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X)$$

for a closed compact C^{∞} manifold X. Let $\text{Diff}^{\mu}(\cdot)$ be the space of differential operators of order μ on the smooth manifold in parentheses, with smooth coefficients (in its natural Fréchet topology). Operators on M close to v are expressed in the splitting of variables $(r, x) \in \mathbb{R}_+ \times X =: X^{\wedge}$. The space $\text{Diff}^{\mu}_{\text{deg}}(M)$ is then defined to be the set of all $A \in \text{Diff}^{\mu}(M \setminus \{v\})$ that are locally near v in the splitting of variables (r, x) of the form

$$(0.2) \qquad A = r^{-\mu} \sum_{j=0}^{\mu} a_j(r)(-r\partial_r)^j$$

for coefficients $a_j \in C^{\infty}(\overline{\mathbb{R}}_+, \text{Diff}^{\mu-j}(X))$. The first component of the pair $\sigma(A) = (\sigma_0(A), \sigma_1(A))$ of principal symbols is the standard homogeneous principal symbol (the interior symbol) of A as an operator in $\text{Diff}^{\mu}(M \setminus \{v\})$, and $\sigma_1(A)$, the conormal symbol of A, is defined as the z-dependent operator family

$$(0.3) \qquad \sigma_1(A))(z) = \sum_{j=0}^{\mu} a_j(0)z^j : H^s(X) \to H^{s-\mu}(X)$$

between standard Sobolev spaces on X. The ellipticity of A with respect to $\sigma(A) = (\sigma_0(A), \sigma_1(A))$ for a fixed weight $\gamma \in \mathbb{R}$ is defined as follows: The interior symbol $\sigma_0(A)$ never vanishes on $T^*(s_0(M)) \setminus 0$, and locally near v in the variables $(r, x) \in \mathbb{R}_+ \times X$ with the covariables (ϱ, ξ) we have

$$(0.4) \qquad \tilde{\sigma}_0(A)(r, x, \varrho, \xi) := r^{\mu} \sigma_0(A)(r, x, r^{-1}\varrho, \xi) \neq 0$$

for $(\varrho, \xi) \neq 0$, up to $r = 0$; moreover, (0.3) is bijective for all $z \in \Gamma_{(n+1)/2-\gamma}$ and any $s \in \mathbb{R}$. Here $n := \dim X$, and

$$(0.5) \qquad \Gamma_{\beta} := \{z \in \mathbb{C} : \text{Re}\, z = \beta\}.$$

The function $\tilde{\sigma}_0(A)$ is also referred to as the rescaled interior symbol of A. As is well-known if A is elliptic with respect to σ_0 the space $L_{\text{cl}}^{-\mu}(s_0(M))$ of classical pseudo-differential operators of order $-\mu$ contains elements P such that $G := PA - 1$ and $C := AP - 1$ have kernels in $C^{\infty}(s_0(M) \times s_0(M))$. Moreover, under the ellipticity conditions with respect to $(\sigma_0(A), \sigma_1(A))$ we find P in the cone algebra, cf. [20] and Section 1.2 below. In this case the left-over terms

are so-called Green operators with discrete asymptotics. From P we may derive
asymptotics of solutions, cf. notation in Section 1.2. We also obtain regularity
in weighted Sobolev spaces over $s_0(M)$. In this article we construct asymptotic
parametrices; those deliver the asymptotic information by a successive process, cf.
Section 1.3. The advantage is that the computation of asymptotics becomes more
transparent and manageable, cf. [6]. Analogous constructions are useful for edge
singularities and singularities of higher order. The edge case will be studied in Sec-
tion 2. Higher order singularities occur in many-particle systems. Corresponding
asymptotic parametrices are interesting as well; the construction is planned in a
forthcoming paper. In any case, the techniques for corner and edge singularities
are crucial for the iterative concept for treating higher order singularities. It turns
out that it is essential to interpret the (degenerate) operators in question as el-
ements of operator algebras, in particular, to consider compositions. Then it is
necessary to admit from the very beginning also higher order operators (and of
negative order when we express parametrices of differential operators), even if we
are finally interested in applications in some specific equation, say, of second order.
Some elements of the pseudo-differential machinery will be formulated below, but
the nature of edge degeneration and of the principal symbolic structure can be
illustrated for differential operators. A manifold M with smooth edge $s_1(M)$ of
dimension $q > 0$ is locally near any point of the edge modelled on

$$(0.6) \qquad\qquad X^\Delta \times \Omega$$

where $\Omega \subseteq \mathbb{R}^q$ corresponds to a chart on $s_1(M)$ with local coordinates y, and X is
a closed compact C^∞ manifold. Then $\mathrm{Diff}^\mu_{\mathrm{deg}}(M)$ for a manifold M with smooth
edge Y denotes the space of those $A \in \mathrm{Diff}^\mu(s_0(M))$ that are locally close to the
edge in the splitting of variables $(r, x, y) \in X^\wedge \times \Omega$ of the form

$$(0.7) \qquad\qquad A = r^{-\mu} \sum_{j+|\alpha| \le \mu} a_{j\alpha}(r, y)(-r\partial_r)^j (rD_y)^\alpha$$

for coefficients $a_{j\alpha} \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, \mathrm{Diff}^{\mu-(j+|\alpha|)}(X))$, see the article [18], or the
monographs [20], [4], [21]. Parallel to the stratification $M = s_0(M) \cup s_1(M)$ as
a disjoint union of smooth manifolds the operators $\mathrm{Diff}^\mu_{\mathrm{deg}}(M)$ have a principal
symbolic structure $\sigma(A) = (\sigma_0(A), \sigma_1(A))$ where $\sigma_0(A) \in C^\infty(T^*(s_0(M)) \setminus 0)$ is
the standard homogeneous principal symbol of A as an operator of $\mathrm{Diff}^\mu(s_0(M))$,
while
$$(0.8)$$
$$\sigma_1(A)(y, \eta) := r^{-\mu} \sum_{j+|\alpha| \le \mu} a_{j\alpha}(0, y)(-r\partial_r)^j (r\eta)^\alpha : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)$$

is operator-valued, acting as a family of operators on the open stretched cone $X^\wedge$,
parametrised by $(y, \eta) \in T^*(s_1(M)) \setminus 0$ (the cotangent bundle of $s_1(M)$ minus
the zero section). The definition of the weighted "Kegel"-spaces $\mathcal{K}^{s,\gamma}(X^\wedge)$ will
be repeated below, cf. Section 2.4. The ellipticity of A with respect to $\sigma(A) = (\sigma_0(A), \sigma_1(A))$ for a fixed weight $\gamma \in \mathbb{R}$ is defined as follows: The interior symbol

$\sigma_0(A)$ never vanishes on $T^*(s_0(M)) \setminus 0$, and locally near $s_1(M)$ in the variables $(r, x, y) \in \mathbb{R}_+ \times X \times \Omega$ with the covariables (ϱ, ξ, η) the rescaled symbol

$$(0.9) \qquad \tilde{\sigma}_0(A)(r, x, y, \varrho, \xi, \eta) := r^\mu \sigma_0(A)(r, x, y, r^{-1}\varrho, \xi, r^{-1}\eta)$$

does not vanish for $(\varrho, \xi, \eta) \neq 0$, up to $r = 0$; moreover, (0.8) is bijective for all $(y, \eta) \in T^*Y \setminus 0$ and any $s \in \mathbb{R}$. It is well-known that the ellipticity condition with respect to $\sigma_0(A)$ entails the Fredholm property of (0.8) for every γ off some discrete set $D(y)$ of reals. In this case, when a certain topological obstruction vanishes (which is the case in our applications) we can enlarge (0.8) by finite rank operators to a bijective 2×2 block matrix family with $\sigma_1(A)(y, \eta)$ in the upper left corner. These extra entries encode trace and potential operators on the level of edge symbols, to guarantee the existence of local parametrices, cf. also Remark 2.14 below. For simplicity we assume for the moment that (0.8) itself is bijective, without such additional data. It may be difficult to explicitly decide for which weights γ this is the case (by virtue of relative index results under changing γ the index of (0.8) can be modified). Concrete results on the index of elliptic edge symbols may be found in [10]. Bijectivity results are also announced in [7]. Another useful observation is that there are always so-called smoothing Mellin plus Green symbols $\sigma_1(M + G)(y, \eta)$ of the edge calculus such that $\sigma_1(A + M + G)(y, \eta)$ is bijective. Below in our constructions we will say more about such structures; let us only note at this point that the bijectivity does not depend on s but depends on $(y, \eta/|\eta|)$. In our construction of asymptotic parametrices under such an ellipticity condition we will refer to the above-mentioned discrete exceptional weights that are determined by the non-bijectivity points of the subordinate conormal symbol

$$(0.10) \qquad \sigma_1(\sigma_1(A))(y, z) := \sum_{j=0}^\mu a_{j0}(0, y)z^j : H^s(X) \to H^{s-\mu}(X)$$

pointwise computed for $\sigma_1(A))(y, \eta) \in C^\infty(T^*\Omega \setminus 0, \text{Diff}^\mu_{\text{deg}}(X^\wedge))$, according to (0.3). Later on, in order to avoid confusion with the different σ_1 we also set $\sigma_c(A)(y, z) := \sigma_1(\sigma_1(A))(y, z)$.
The main intention of this paper is to develop a new concise approach to pseudo-differential cone and edge algebras and to elliptic regularity with asymptotics. The recent experience of the authors with the problem of explictly computing the asymptotic data of solutions, cf. the articles [6], [7], is that it needs parallel work to organise the construction of parametrices and of the underlying symbolic structures in a more transparent and accessible way. Of course, we refer to the basics of the cone and edge calculus, known from a number of articles and monographs, cf. [18], [20], [4], [21]. Already several typical applications to the crack theory or to mixed problems required once again voluminous extensions, cf. [11] or [10]. The concrete models that we have in mind here and in future should be understandable in terms of a minimal theoretical machinery. The point is that in particle physics relatively simple elliptic operators (essentially Laplacians on a smooth manifold, e.g. in $\mathbb{R}^{3N}$) are combined with singular potentials of the type

$\sum_{j,k\leq N} c_{jk}|\mathbf{x^j} - \mathbf{x^k}|^{-1}$ where $\mathbf{x^j} = (x_1^j, x_2^j, x_3^j)$ represents the position of the j-th particle in $\mathbb{R}^3$. The singularities of the potential give rise to a corner manifold, embedded in $\mathbb{R}^{3N}$, and the problem is to understand the asymptotics of solutions close to that singular set. This is a task of "unexpected" complexity, even for small $N (= 1$ or $2)$, corresponding to conical or edge singularities. We found it therefore justified to reorganise the way of deriving asymptotics of solutions by means of so-called asymptotic parametrices which are introduced in this paper both for the conical and the edge case. Since the concept is based on "ordinary" parametrix constructions we also discuss this aspect here, either with new proofs or corresponding references.

This paper is organised as follows. In Section 1 we formulate the cone algebra including its symbolic structure, ellipticity and (asymptotic) parametrices. This material may also be regarded as a tool and simplest model for the case of edge singularities in Section 2 which is the main part of this article. We establish the edge algebra with discrete asymptotics, including the symbolic machinery and the structure of ellipticity with (asymptotic) parametrices. In addition we discuss several special cases and simplifications of the general approach, e.g. when (as in our applications to the above-mentioned case, to be published elsewhere) the original operator is a differential operator. Let us finally note that larger N requires the calculus for higher order corners. Also this technique should be further smoothened and simplified, to make the case of higher particle numbers accessible.

1. Elliptic cone operators

1.1. Operators of Fuchs type

Let M be a manifold with conical singularity $v := s_1(M)$. Because of (0.1) we have a local identification of $M \setminus \{v\}$ with $X^\wedge$ which allows us to attach a copy of X to $M \setminus \{v\}$. In this way we get the stretched manifold $\mathbb{M}$ associated with M which is a smooth manifold with boundary $\partial\mathbb{M} \cong X$. A collar neighbourhood of $\partial\mathbb{M}$ may be identified with $\overline{\mathbb{R}}_+ \times X$ (or, equivalently, with $[0,1) \times X$). For purposes below on $M \setminus \{v\}$ we fix a strictly positive function $\mathbf{r}^1 \in C^\infty(M \setminus \{v\})$ that is equal to r close to v in a fixed splitting of variables $(r, x) \in \mathbb{R}_+ \times X$. For any $\beta \in \mathbb{R}$ we call $\mathbf{r}^\beta$ a weight function with weight β.

The cone algebra on M is defined as a union of subspaces $L^\mu(M, \mathbf{g}) \subset L^\mu_{\mathrm{cl}}(M \setminus \{v\}), \mu \in \mathbb{R}$, labelled with weight data $\mathbf{g} := (\gamma, \gamma - \mu, \Theta)$ for a weight $\gamma \in \mathbb{R}$ and a weight interval Θ, to be specified below. Here $L^\mu_{\mathrm{cl}}(\cdot)$ is the space of classical pseudo-differential operators on the manifold in parentheses (assumed to be Riemannian, with an asociated metric dx). We also employ the parameter-dependent variant $L^\mu_{\mathrm{cl}}(\cdot; \mathbb{R}^l), \lambda \in \mathbb{R}^l$, i.e. the space of families of operators $A(\lambda)$ in $L^\mu_{\mathrm{cl}}(\cdot)$, modulo $L^{-\infty}(\cdot; \mathbb{R}^l) := \mathcal{S}(\mathbb{R}^l, L^{-\infty}(\cdot))$ described by local amplitude functions $a(x, \xi, \lambda)$ with (ξ, λ) being treated as covariables. One of the ingredients of the cone algebra is the space $M^\mu_\mathcal{O}(X)$ of holomorphic $L^\mu_{\mathrm{cl}}(X)$-valued Mellin symbols. The background

is that the operator $-r\partial_r$ occurring in (0.2) can be expressed as a Mellin pseudo-differential operator on the half-axis with the symbol z, namely, $-r\partial_r = M^{-1}zM$, say, as an operator $C_0^\infty(\mathbb{R}_+) \to C_0^\infty(\mathbb{R}_+)$. Here

$$(1.1) \quad Mu(z) = \int_0^\infty r^z u(r)dr/r \text{ with the inverse } M^{-1}g(r) = \int_{\Gamma_{1/2-\gamma}} r^{-z}g(z)đz$$

for any $\gamma \in \mathbb{R}, đz := (2\pi i)^{-1}dz$.

In the edge calculus below we need Mellin symbols also in parameter-dependent form. Let us recall the definition. First, if $U \subseteq \mathbb{C}$ is an open set and E a Fréchet space, by $\mathcal{A}(U, E)$ we denote the space of holomorphic functions in U with values in E in the Fréchet topology of uniform convergence on compact sets. The space $M_\mathcal{O}^\mu(X; \mathbb{R}^l)$ is defined to be the set of all $h \in \mathcal{A}(\mathbb{C}, L_{cl}^\mu(X; \mathbb{R}^l))$ such that $h(z, \lambda)|_{\Gamma_\beta \times \mathbb{R}^l} \in L_{cl}^\mu(X; \Gamma_\beta \times \mathbb{R}^l))$ for every real β, uniformly in compact β-intervals. In the latter notation $\operatorname{Re} z$ for $z \in \Gamma_\beta$ belongs to the parameters.

For a Mellin amplitude function $f(r, r', z, \lambda)$ taking values in $L_{cl}^\mu(X; \Gamma_{1/2-\gamma} \times \mathbb{R}^l)$ we form the associated (parameter-dependent) weighted Mellin pseudo-differential operator

(1.2)
$$\operatorname{op}_M^\gamma(f)(\lambda)u(r) = \int_\mathbb{R} \int_0^\infty (r/r')^{-(1/2-\gamma+i\rho)} f(r, r', 1/2 - \gamma + i\rho, \lambda)u(r')dr'/r'đ\rho,$$

$đ\rho = (2\pi)^{-1}d\rho$, interpreted as a Mellin oscillatory integral and representing a family of operators

$$(1.3) \qquad\qquad C_0^\infty(\mathbb{R}_+, C^\infty(X)) \to C^\infty(\mathbb{R}_+, C^\infty(X)).$$

in $L_{cl}^\mu(X^\wedge; \mathbb{R}^l)$. We can also write

$$\operatorname{op}_M^\gamma(f)u = r^\gamma \operatorname{op}_M(T^{-\gamma}f)r^{-\gamma}u$$

for $\operatorname{op}_M(\cdot) := \operatorname{op}_M^0(\cdot)$ and $(T^\beta f)(z) := f(z + \beta), \beta \in \mathbb{R}$ (the other variables in the latter expression have been suppressed).

Theorem 1.1. (*cf.* [21, Section 2.2.2]) *For every* $f(z, \lambda) \in L_{cl}^\mu(X; \Gamma_\beta \times \mathbb{R}^l)$ *for some fixed* $\beta \in \mathbb{R}$ *there exists an* $h(z, \lambda) \in M_\mathcal{O}^\mu(X; \mathbb{R}^l)$ *such that*

$$(1.4) \qquad\qquad f = h|_{\Gamma_\beta \times \mathbb{R}^l} \mod L^{-\infty}(X; \Gamma_\beta \times \mathbb{R}^l),$$

and h *is unique modulo* $M_\mathcal{O}^{-\infty}(X; \mathbb{R}^l)$.

Theorem 1.2. (*cf.* [19], *or* [21, Section 2.2.2]) *Let*

$$(1.5) \qquad\qquad \tilde{p}(r, \tilde{\rho}) \in C^\infty(\overline{\mathbb{R}}_+, L_{cl}^\mu(X; \mathbb{R}_{\tilde{\rho}})), \quad p(r, \rho) := \tilde{p}(r, r\rho).$$

Then there is an $h(r, z) \in C^\infty(\overline{\mathbb{R}}_+, M_\mathcal{O}^\mu(X))$, *unique modulo* $C^\infty(\overline{\mathbb{R}}_+, M_\mathcal{O}^{-\infty}(X))$, *such that*

$$(1.6) \qquad\qquad \operatorname{Op}_r(p) = \operatorname{op}_M^\beta(h) \mod L^{-\infty}(X^\wedge)$$

for every $\beta \in \mathbb{R}$. Conversely, for every $h(r,z) \in C^\infty(\overline{\mathbb{R}}_+, M_\mathcal{O}^\mu(X))$ there exists a $\tilde{p}(r, \tilde{\rho}) \in C^\infty(\overline{\mathbb{R}}_+, L_{cl}^\mu(X; \mathbb{R}_{\tilde\rho}))$, unique modulo $C^\infty(\overline{\mathbb{R}}_+, L^{-\infty}(X; \mathbb{R}_{\tilde\rho}))$, such that (1.6) holds for every $\beta \in \mathbb{R}$.

The relation (1.5) refers to pseudo-differential operators on $X^\wedge$ in the sense of mappings (1.3).

The structures of the cone algebra may be deduced by the task to express the parametrix of an elliptic operator of the form (0.2). For $h(r,z) := \sum_{j=0}^\mu a_j(r) z^j$ which is an element of $C^\infty(\overline{\mathbb{R}}_+, M_\mathcal{O}^\mu(X))$ we write

$$(1.7) \qquad A = r^{-\mu} \mathrm{op}_M^{\gamma - n/2}(h)$$

for any fixed $\gamma \in \mathbb{R}$ and $n := \dim X$. In order to briefly recall the result we identify $\mathrm{int}\, M := M \setminus \{v\}$ with $\mathrm{int}\, \mathbb{M} = \mathbb{M} \setminus \partial\mathbb{M}$ and a collar neighbourhood of $\partial\mathbb{M}$ in $\mathbb{M}$ with $\overline{\mathbb{R}}_+ \times X$ in the splitting of variables (r, x). By a cut-off function on $\mathbb{M}$ we understand any real-valued $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ that is equal to 1 in a neighbourhood of $r = 0$. Given two functions $\varphi, \varphi' \in C^\infty(\mathbb{M})$ we write $\varphi \prec \varphi'$ if $\varphi' \equiv 1$ on $\mathrm{supp}\,\varphi$. Now let $\omega, \omega', \omega''$ be cut-off functions with $\omega'' \prec \omega \prec \omega'$, and set $\chi := 1 - \omega, \chi' := 1 - \omega''$. Then every $A \in \mathrm{Diff}_{\mathrm{deg}}^\mu(M)$ can be written in the form

$$(1.8) \qquad A = \omega r^{-\mu} \mathrm{op}_M^{\gamma-n/2}(h) \omega' + \chi A_{\mathrm{int}} \chi'$$

where A_{int} in the second summand on the right is A itself. In general, the pseudo-differential operators of the cone algebra have the form

$$(1.9) \qquad A = \omega r^{-\mu} \mathrm{op}_M^{\gamma-n/2}(h) \omega' + \chi A_{\mathrm{int}} \chi' + M + G$$

for arbitrary $h \in C^\infty(\overline{\mathbb{R}}_+, M_\mathcal{O}^\mu(X))$, $A_{\mathrm{int}} \in L_{cl}^\mu(\mathrm{int}\, M)$, and additional summands $M + G$, so-called smoothing Mellin plus Green operators. The latter represent asymptotic information. In order to explain their structure we formulate weighted cone spaces and subspaces with asymptotics.

First $\mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^n)$ for $s, \gamma \in \mathbb{R}$ is defined as the completion of $C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^n)$ with respect to the norm

$$(1.10) \quad \|u\|_{\mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^n)} := \left\{ \int_{\mathbb{R}^n} \int_{\Gamma_{(n+1)/2-\gamma}} \langle z, \xi \rangle^{2s} |M_{r \to z} F_{x \to \xi} u(z, \xi)|^2 đz d\xi \right\}^{1/2}.$$

For compact X we then obtain the spaces $\mathcal{H}^{s,\gamma}(X^\wedge), X^\wedge = \mathbb{R}_+ \times X$, by using (1.10) as local representations, combined with a partition of unity. The notation with calligraphic letters indicates that the spaces contain some specific information also at $r = \infty$. On a compact manifold M with conical singularity v we set

$$(1.11) \qquad H^{s,\gamma}(M) := \{u \in H_{\mathrm{loc}}^s(M \setminus \{v\}) : \omega u \in \mathcal{H}^{s,\gamma}(X^\wedge)\}$$

for some cut-off function ω. Recall that for convenience we identify $M \setminus \{v\}$ locally close to the singularitiy with $X^\wedge$, i.e. we often suppress pull backs or push forwards under a corresponding map from the local model to the manifold itself. This map is kept fixed, in particular, the local splitting of variables (r, x). This is the usual

point of view when we formulate asymptotics for $r \to 0$. Observe that the operator of multiplication by $\mathbf{r}^\beta, \beta \in \mathbb{R}$, induces an isomorphism

$$(1.12) \qquad \mathbf{r}^\beta : H^{s,\gamma}(M) \to H^{s,\gamma+\beta}(M).$$

Here $\mathbf{r}^\beta$ is the weight function defined at the beginning of this section.

Asymptotics of an element $u \in H^{s,\gamma}(M)$ of type $P := \{(p_j, m_j)\}_{j \in \mathbb{N}} \subset \mathbb{C} \times \mathbb{N}$ (where we assume $\pi_{\mathbb{C}} P := \{p_j\}_{j \in \mathbb{N}} \subset \{\operatorname{Re} z < (n+1)/2 - \gamma\}$ and $\operatorname{Re} p_j \to -\infty$ as $j \to \infty$ when $\pi_{\mathbb{C}} P$ is an infinite set) means that for every $\beta \geq 0$ there is an $N = N(\beta) \in \mathbb{N}$ such that for any cut-off function ω

$$(1.13) \qquad u - \sum_{j=0}^{N} \sum_{k=0}^{m_j} \omega(r) c_{jk} r^{-p_j} \log^k r \in \mathbf{r}^\beta H^{s,\gamma}(M)$$

for some coefficients $c_{jk} \in C^\infty(X)$. It can easily be proved that the coefficients are uniquely determined by u. The space of those u will be denoted by $H_P^{s,\gamma}(M)$. It may also be interesting to control the asymptotics in terms of a condition $c_{jk} \in L_j, 0 \leq k \leq m_j$, where L_j are prescribed finite-dimensional subspaces of $C^\infty(X)$. In this case the asymptotic type is a corresponding sequence of triples $\{(p_j, m_j, L_j)\}_{j \in \mathbb{N}}$. In the general discussion we content ourselves with the case $P := \{(p_j, m_j)\}_{j \in \mathbb{N}}$. Asymptotics including L_j are a simple modification.

For computations it is convenient to take into account also finite asymptotic expansions. We say that a $P = \{(p_j, m_j)\}_{j=0,\dots,J}$ for a $J = J(P) \in \mathbb{N}$ is associated with the weight data $\mathbf{g} := (\gamma, \Theta)$ for a finite (so-called weight interval) $\Theta := (\vartheta, 0] \subset \overline{\mathbb{R}}_-$ if $\pi_{\mathbb{C}} P := \{p_j\}_{j=0,\dots,J} \subset \{(n+1)/2 - \gamma + \vartheta < \operatorname{Re} z < (n+1)/2 - \gamma\}$. An asymptotic type P is said to satisfy the shadow condition if $p \in \pi_{\mathbb{C}} P$ entails $p - j \in \pi_{\mathbb{C}} P$ for all $j \in \mathbb{N}$ with $\operatorname{Re} p - j > (n+1)/2 - \gamma + \vartheta$. Let Θ be finite. The spaces

$$(1.14) \qquad \mathcal{E}_P := \{\sum_{j=0}^{J} \sum_{k=0}^{m_j} \omega(r) c_{jk} r^{-p_j} \log^k : c_{jk} \in C^\infty(X) \text{ for all } j, k\}$$

and

$$(1.15) \qquad H_\Theta^{s,\gamma}(M) := \bigcap_{\varepsilon > 0} H^{s,\gamma-\vartheta-\varepsilon}(M)$$

are both Fréchet in a natural way, and we have $H_\Theta^{s,\gamma}(M) \cap \mathcal{E}_P = \{0\}$. We set

$$(1.16) \qquad H_P^{s,\gamma}(M) := \mathcal{E}_P + H_\Theta^{s,\gamma}(M)$$

in the Fréchet topology of the direct sum. Note that in the above case $P := \{(p_j, m_j)\}_{j \in \mathbb{N}} \subset \mathbb{C} \times \mathbb{N}$ corresponds to $\Theta = (-\infty, 0]$, and the space $H_P^{s,\gamma}(M)$ can equivalently be defined as $H_P^{s,\gamma}(M) := \bigcap_{k \in \mathbb{N}} H_{P_k}^{s,\gamma}(M)$ where $P_k := \{(p,m) \in P : \operatorname{Re} p_k > (n+1)/2 - \gamma - (k+1)\}$ is finite, associated with the weight data (γ, Θ_k) for $\Theta_k := (-(k+1), 0], k \in \mathbb{N}$.

An operator $G \in \bigcap_{s \in \mathbb{R}} \mathcal{L}(H^{s,\gamma}(M), H^{\infty,\gamma-\mu}(M))$ is said to be a Green operator in the cone algebra, associated with the weight data $\mathbf{g} := (\gamma, \gamma - \mu, \Theta)$ if it induces continuous operators

(1.17) $G : H^{s,\gamma}(M) \to H_P^{\infty,\gamma-\mu}(M), \ \ G^* : H^{s,-\gamma+\mu}(M) \to H_Q^{\infty,-\gamma}(M),$

for all $s \in \mathbb{R}$ and G-dependent asymptotic types P, Q, associated with the weight data $(\gamma - \mu, \Theta)$ and $(-\gamma, \Theta)$, respectively. By

$$L_G(M, \mathbf{g}) \text{ for } \mathbf{g} := (\gamma, \gamma - \mu, \Theta)$$

we denote the space of all Green operators. Let us also recall the notion of smoothing Mellin operators, occurring in (1.9). A sequence

(1.18) $R := \{(r_j, n_j)\}_{j \in \mathbb{Z}} \subset \mathbb{C} \times \mathbb{N} \text{ with } |\operatorname{Re} r_j| \to \infty \text{ as } |j| \to \infty$

is called a Mellin asymptotic type. Set $\pi_\mathbb{C} R := \{r_j\}_{j \in \mathbb{Z}}$. A $\chi \in C^\infty(\mathbb{C})$ is called a $\pi_\mathbb{C} R$-excision function if $\chi(z) = 0$ for $\operatorname{dist}(z, \pi_\mathbb{C} R) < \varepsilon_0$, and $\chi(z) = 1$ for $\operatorname{dist}(z, \pi_\mathbb{C} R) > \varepsilon_1$ for some $0 < \varepsilon_0 < \varepsilon_1$. The space $M_R^{-\infty}(X)$ is defined to be the set of all $f \in \mathcal{A}(\mathbb{C} \setminus \pi_\mathbb{C} R, L^{-\infty}(X))$ such that $\chi f|_{\Gamma_\beta} \in L^{-\infty}(X; \Gamma_\beta)$ for every $\beta \in \mathbb{R}$ and any $\pi_\mathbb{C} R$-excision function χ, uniformly in finite β-intervals, and in addition f is meromorphic with poles at the points r_j of multiplicity $n_j + 1$ and Laurent coefficients at $(z - r_j)^{-(k+1)}, 0 \le k \le n_j$, in $L^{-\infty}(X)$ of finite rank. Moreover, we set

(1.19) $$M_R^\mu(X) := M_\mathcal{O}^\mu(X) + M_R^{-\infty}(X).$$

Remark 1.3. The decomposition (1.19) is not direct since $M_\mathcal{O}^\mu(X) \cap M_R^{-\infty}(X) = M_\mathcal{O}^{-\infty}(X)$ is non-trivial. For any $f \in M_R^\mu(X)$ we can produce a $g \in M_R^{-\infty}(X)$ such that $f - g \in M_\mathcal{O}^\mu(X)$ by applying Theorem 1.1 to $f|_{\Gamma_\beta} \in L_{\mathrm{cl}}^\mu(X; \Gamma_\beta)$ for any $\beta \in \mathbb{R}$ with $\pi_\mathbb{C} R \cap \Gamma_\beta = \emptyset$. More generally, this process applies to $f(y, z) \in C^\infty(\Omega, M_R^\mu(X)), \Omega \subseteq \mathbb{R}^q$ open, and gives us a $g(y, z) \in C^\infty(\Omega, M_R^{-\infty}(X))$ such that $f(y, z) - g(y, z) \in C^\infty(\Omega, M_\mathcal{O}^\mu(X))$.

Remark 1.4. Let $f \in M_R^\mu(X)$ and choose a $\beta \in \mathbb{R}$ such that $\pi_\mathbb{C} R \cap \Gamma_\beta = \emptyset$, and assume that $f|_{\Gamma_\beta} \in L_{\mathrm{cl}}^{\mu-1}(X; \Gamma_\beta)$. Then we have $f \in M_R^{\mu-1}(X)$.

Theorem 1.5. (*cf.* [20, Section 2.1.2]) *We have*

(1.20) $f \in M_R^\mu(X), g \in M_Q^\nu(X) \Rightarrow fg \in M_S^{\mu+\nu}(X)$

for every R, Q with some resulting S.

The interpretation of the multiplication in Theorem 1.5 is z-wise, first for all $z \in \mathbb{C} \setminus (\pi_\mathbb{C} R \cup \pi_\mathbb{C} Q)$; then this product extends across $\pi_\mathbb{C} R \cup \pi_\mathbb{C} Q$ to an element of $M_S^{\mu+\nu}(X)$. In particular, if $f \in M_R^{-\infty}(X)$ or $f \in M_Q^{-\infty}(X)$, then $fg \in M_S^{-\infty}(X)$.

Theorem 1.6. (*cf.* [21, Section 2.4.3]) *For every $m \in M_R^{-\infty}(X)$ for some Mellin asymptotic type R there exists an $l \in M_S^{-\infty}(X)$ for another Mellin asymptotic type S such that*

(1.21) $$(1 + m)^{-1} = 1 + l,$$

i.e. $(1+m)(1+l) = (1+l)(1+m) = 1$ *in the sense of the multiplication of the preceding theorem.*

More generally we have the following result.

Theorem 1.7. *Let $f \in M_R^\mu(X)$ and assume that $f|_{\Gamma_\beta} \in L_{cl}^\mu(X, \Gamma_\beta)$ is parameter-dependent elliptic. Then we have $f^{-1} \in M_S^{-\mu}(X)$ for some Mellin asymptotic type S.*

Proof. Let $g_\beta \in L_{cl}^{-\mu}(X, \Gamma_\beta)$ be a parameter-dependent parametrix of $f|_{\Gamma_\beta}$. Applying Theorem 1.1 for $l = 0$ we find an $h \in M_{\mathcal{O}}^{-\mu}(X)$ such that $g_\beta = h|_{\Gamma_\beta}$ modulo $L^{-\infty}(X; \Gamma_\beta)$. It follows that

$$(hf)|_{\Gamma_\beta} = 1 + r_\beta \text{ for some } r_\beta \in L^{-\infty}(X; \Gamma_\beta).$$

However, then we know that $hf = 1 + m$ for some $m \in M_R^{-\infty}(X)$. From Theorem 1.6 it follows that $(1+l)hf = 1$, i.e. $(1+l)h = f^{-1}$ is as claimed. $\qquad\square$

A smoothing Mellin operator in the cone calculus associated with the weight data $\mathbf{g} := (\gamma, \gamma - \mu, \Theta)$ for $\Theta = (-(k+1), 0], k \in \mathbb{N}$, is defined as

$$(1.22) \qquad M := r^{-\mu} \sum_{j=0}^{k} \omega r^j \operatorname{op}_M^{\gamma_j - n/2}(f_j)\omega'$$

for cut-off functions ω, ω', elements $f_j \in M_{R_j}^{-\infty}(X)$ for Mellin asymptotic types R_j, and weights γ_j with $\gamma - j \le \gamma_j \le \gamma$ and $\pi_{\mathbb{C}} R_j \cap \Gamma_{(n+1)/2 - \gamma_j} = \emptyset$ for all j. Let

$$L_{M+G}(M, \mathbf{g}) \text{ for } \mathbf{g} := (\gamma, \gamma - \mu, \Theta), \ \Theta = (-(k+1), 0],$$

denote the space of all operators $M + G$ for arbitrary $G \in L_G(M, \mathbf{g})$ and operators (1.22) for arbitrary f_j, $j = 0, \ldots, k$, for any fixed choice of cut-off functions ω, ω'.

Now we have all ingredients of (1.9). Let

$$(1.23) \qquad L^\mu(M, \mathbf{g}) \text{ for } \mathbf{g} = (\gamma, \gamma - \mu, \Theta), \ \Theta = (-(k+1), 0],$$

denote the space of all operators of the form (1.9) for arbitrary $h(r, z) \in C^\infty(\overline{\mathbb{R}}_+, M_{\mathcal{O}}^\mu(X))$, $A_{int} \in L_{cl}^\mu(s_0(M))$, and $M + G \in L_{M+G}(M, \mathbf{g})$. For $\Theta = (-\infty, 0]$ we set $L^\mu(M, \mathbf{g}) := \bigcap_{k \in \mathbb{N}} L^\mu(M, (\gamma, \gamma - \mu, (-(k+1), 0]))$. These spaces constitute the cone algebra over the compact manifold M with conical singularity v. The cone algebra in this form has been introduced in [16].

Remark 1.8. The space $\operatorname{Diff}_{\deg}^\mu(M)$ belongs to $L^\mu(M, \mathbf{g})$ for every weight γ and $\Theta = (-\infty, 0]$.

Remark 1.9. We have

$$(1.24) \qquad L^\mu(M, \mathbf{g}) \cap L^{-\infty}(s_0(M)) = L_{M+G}(M, \mathbf{g}).$$

In fact, if A belongs to the left hand side of (1.24) then all parameter-dependent homogeneous components of the local (over X) symbol of $h(r, z) \in C^\infty(\overline{\mathbb{R}}_+, L^\mu_{\mathrm{cl}}(X; \Gamma_{(n+1)/2-\gamma}))$ vanish for $r > 0$. Since those are smooth up to $r = 0$ they vanish up to 0, and it follows that $h(r, z) \in C^\infty(\overline{\mathbb{R}}_+, L^{-\infty}(X; \Gamma_{(n+1)/2-\gamma}))$, more precisely, $h(r, z) \in C^\infty(\overline{\mathbb{R}}_+, M^{-\infty}_{\mathcal{O}}(X))$, and for such an h we have $\omega r^{-\mu} \mathrm{op}_M^{\gamma-n/2}(h)\omega' \in L_{M+G}(M, \mathbf{g})$. Every $A \in L^\mu(M, \mathbf{g})$ induces continuous operators

$$(1.25) \qquad A : H^{s,\gamma}(M) \to H^{s-\mu,\gamma-\mu}(M), \ H^{s,\gamma}_P(M) \to H^{s-\mu,\gamma-\mu}_Q(M)$$

for every $s \in \mathbb{R}$ and every asymptotic type P associated with the weight data (γ, Θ) for some resulting Q associated with the weight data $(\gamma - \mu, \Theta)$. More details may be found, in [20], see also [21].

The composition refers to a bilinear operation
$$(1.26)$$
$$L^\mu(M, (\gamma-\nu, \gamma-(\mu+\nu), \Theta)) \times L^\nu(M, (\gamma, \gamma-\nu, \Theta)) \to L^{\mu+\nu}(M, (\gamma, \gamma-(\mu+\nu), \Theta))$$

Recall, in particular, that the spaces

$$(1.27) \qquad\qquad L_{M+G}(M, \mathbf{g}) \ \text{and} \ L_G(M, \mathbf{g})$$

of smoothing Mellin plus Green operators $M+G$, and G, respectively, form subalgebras, where AB belongs to L_{M+G} (L_G) as soon as A or B belongs to L_{M+G} (L_G). We have also a formal adjoint

$$(1.28) \qquad L^\mu(M, (\gamma, \gamma-\mu, \Theta)) \to L^\mu(M, (-\gamma+\mu, -\gamma, \Theta))$$

defined by $(Au, v)_{H^{0,0}(M)} = (u, A^*v)_{H^{0,0}(M)}$ for all $u, v \in C_0^\infty(M \setminus \{v\})$, and L_{M+G} (L_G) are preserved under formal adjoints.

Let us now express the principal symbolic structure $\sigma(A) = (\sigma_0(A), \sigma_1(A))$ of operators $A \in L^\mu(M, (\gamma, \gamma-\mu, \Theta))$. The relation

$$L^\mu(M, (\gamma, \gamma-\mu, \Theta)) \subset L^\mu_{\mathrm{cl}}(M \setminus \{v\})$$

gives rise to the standard homogeneous principal symbol $\sigma_0(A)$ of order μ of the operator as an invariantly defined function in $C^\infty(T^*(s_0(M))\backslash 0)$, $s_0(M) = M\backslash\{v\}$. In addition, as noted in the introduction we have the rescaled principal symbol (0.9) close to $s_1(M) = v$. The principal conormal symbol $\sigma_1(A)$ is defined as the operator function

$$(1.29) \qquad \sigma_1(A)(z) := h(0, z) + f_0(z) : H^s(X) \to H^{s-\mu}(X)$$

for $z \in \Gamma_{(n+1)/2-\gamma}$, cf. the notation in connection with (1.9) and (1.22). The behaviour of $\sigma(\cdot)$ under compositions $(A, B) \to AB$ in the notation of (1.26) is as follows:

$$(1.30) \qquad \sigma_0(AB) = \sigma_0(A)\sigma_0(B), \ \sigma_1(AB) = (T^\nu \sigma_1(A))\sigma_1(B)$$

where $(T^\beta f)(z) = f(z + \beta)$.

Let us also recall the Mellin translation product, cf. [21], referring to the lower order conormal symbols

$$(\sigma_1^{\mu-j}(A)(z))_{j=0,1,2,\dots} \text{ for } \sigma_1^{\mu-j}(A)(z) := 1/j!(\partial_r^j h)(0,z) + f_j(z),$$

$\sigma_1^{\mu}(A)(z) := \sigma_1(A)(z).$

Remark 1.10. We have

(1.31) $$\sigma_1^{\mu+\nu-l}(AB) = \sum_{i+j=l}(T^{\nu-j}\sigma_1^{\mu-i}(A))\sigma_1^{\nu-j}(B),$$

$l = 0,1,2,\dots.$

1.2. Ellipticity in the cone algebra

Definition 1.11. Let $A \in L^{\mu}(M,\mathbf{g}), \mathbf{g} = (\gamma, \gamma-\mu, \Theta).$

  (i) The operator A is called σ_0-elliptic, if $\sigma_0(A)$ never vanishes on $T^*(s_0(M))\setminus$ 0, and if in addition close to $s_1(M)$ we have $\tilde{\sigma}_0(A)(r,x,\rho,\xi) \neq 0$ for $(\rho,\xi) \neq 0$, up to $r = 0$.
  (ii) The operator A is called σ_1-elliptic if (1.29) is a family of isomorphisms for all $z \in \Gamma_{(n+1)/2-\gamma}$ for any $s \in \mathbb{R}$.

We call A elliptic if it satisfies the conditions (i), (ii).

Remark 1.12. If A is elliptic we find a $\tilde{p}(r,\tilde{\rho}) \in C^{\infty}(\overline{\mathbb{R}}_+, L_{cl}^{\mu}(X;\mathbb{R}_{\tilde{\rho}})$ with values in parameter-dependent elliptic operators on X (with parameter $\tilde{\rho}$) such that $h(r,z) \in C^{\infty}(\overline{\mathbb{R}}_+, M_{\mathcal{O}}^{\mu}(X))$ in the representation (1.9) is associated with $\tilde{p}$ via Theorem 1.2.

Remark 1.13. It may be difficult to explicitly check the condition (ii) of Definition 1.11; the non-bijectivity points of (1.29) in the complex plane are a kind of non-linear eigenvalues of $\sigma_1(A)$ as operators globally on X. Often in applications we have $A \in \text{Diff}_{\text{deg}}^{\mu}(M)$, and X may have several connected components $X_1, \dots, X_N$. In this case, because of the locality of differential operators, $\sigma_1(A)(z) : H^s(X) \to H^{s-\mu}(X)$ is bijective if and only if $\sigma_1(A)(z) : H^s(X_j) \to H^{s-\mu}(X_j)$ is bijective for every $j = 1, \dots, N$.

Theorem 1.14. *Let M be a manifold with conical singularities. An elliptic operator $A \in L^{\mu}(M,\mathbf{g})$ for $\mathbf{g} = (\gamma, \gamma-\mu, \Theta)$ has a parametrix $P \in L^{-\mu}(M,\mathbf{g}^{-1})$ for $\mathbf{g}^{-1} = (\gamma-\mu, \gamma, \Theta)$ in the sense that $PA - 1 \in L_G(M,\mathbf{g}_l), AP - 1 \in L_G(M,\mathbf{g}_r)$ for $\mathbf{g}_l = (\gamma, \gamma, \Theta), \mathbf{g}_r = (\gamma-\mu, \gamma-\mu, \Theta).$*

Proof. Applying Remark 1.12 we choose a $\tilde{p}(r,\tilde{\rho}) \in C^{\infty}(\overline{\mathbb{R}}_+, L_{cl}^{\mu}(X;\mathbb{R}_{\tilde{\rho}}))$ with values in parameter-dependent elliptic operators, and set $p(r,\rho) := \tilde{p}(r,r\rho)$. Near the conical point in the spitting of variables $(r,x) \in X^{\wedge}$ we have $\omega A \omega' = \omega r^{-\mu}\text{Op}_r(p)\omega'$ modulo $L^{-\infty}(X^{\wedge})$. Let us choose an operator function $\tilde{p}^{(-1)}(r,\tilde{\rho}) \in C^{\infty}(\overline{\mathbb{R}}_+, L_{cl}^{-\mu}(X;\mathbb{R}_{\tilde{\rho}}))$ such that $p^{(-1)}(r,\rho) := \tilde{p}^{(-1)}(r,r\rho)$ satisfies the relations

$$r^{\mu}p^{(-1)}(r,\rho)\sharp_r r^{-\mu}p(r,\rho) \sim 1, \quad r^{-\mu}p(r,\rho)\sharp_r r^{\mu}p^{(-1)}(r,\rho) \sim 1$$

where $\sharp_r$ means the Leibniz product between the respective symbols (cf. also Lemma 2.20 below). By virtue of Theorem 1.2 there is an

$$h^{(-1)}(r, z) \in C^\infty(\overline{\mathbb{R}}_+, M_\mathcal{O}^{-\mu}(X))$$

such that $\omega r^\mu \mathrm{Op}_r(p^{(-1)})\omega' = \omega r^\mu \mathrm{op}_M^{\gamma-\mu-n/2}(h^{(-1)})\omega'$ modulo a smoothing remainder.

Without loss of generality in the representation (1.9) we may assume that A_{int} coincides with A. Then A_{int} is elliptic on $s_0(M)$ in the standard sense, and we find a parametrix $A_{\mathrm{int}}^{(-1)} \in L_{\mathrm{cl}}^{-\mu}(s_0(M))$. Now we set $A^{(-1)} := r^\mu \omega \mathrm{op}_M^{\gamma-\mu-n/2}(h^{(-1)})\omega' + \chi A_{\mathrm{int}}^{(-1)}\chi'$. Then we have $A^{(-1)} \in L^{-\mu}(M, \mathbf{g}^{-1})$ and $A^{(-1)}A - 1 \in L^0(M, \mathbf{g}_l) \cap L^{-\infty}(s_0(M))$. Thus by virtue of Remark 1.9 we have $A^{(-1)}A - 1 \in L_{M+G}(M, \mathbf{g}_l)$. Next we construct an operator $B \in L_{M+G}(M, \mathbf{g}_l)$ such that

$$(1.32) \qquad\qquad (A^{(-1)} + B)A = 1 \bmod L_G(M, \mathbf{g}_l).$$

We first find an $l \in M_R^{-\infty}(X)$ for some Mellin asymptotic type R such that

$$(1.33) \qquad\qquad (h^{-1}(0, z+\mu) + l(z+\mu))(h(0, z) + f_0(z)) = 1,$$

cf. the notation in (1.22). We use the fact that $(h(0, z) + f_0(z))^{-1} = h^{-1}(0, z + \mu) + m(z + \mu)$ for some $m(z) \in M_S^{-\infty}(X)$, cf. Theorem 1.7. Thus it suffices to set $l(z) = m(z)$. For $B_0 := r^\mu \omega \mathrm{op}_M^{\gamma-\mu-n/2}(l)\omega'$ we obtain

$$(A^{(-1)} + B_0)A = 1 + C \text{ for a } C \in L_{M+G}(M, \mathbf{g}_l)$$

and $\sigma_1(C) = 0$. Now we use the fact that $(\sum_{j=0}^k (-1)^j C^j)(1+C) = 1+G$ for some $G \in L_G(M, \mathbf{g})$. It follows that

$$(1.34) \qquad\qquad \left(\sum_{j=0}^k (-1)^j C^j\right)(A^{(-1)} + B_0) =: P$$

is a left parametrix of A. In other words, we have constructed $B := (\sum_{j=0}^k (-1)^j C^j) B_0 + (\sum_{j=1}^k (-1)^j C^j)A^{(-1)}$ mentioned before in (1.32). In an analogous manner we obtain a right parametrix with a left-over term in $L_G(M, \mathbf{g}_r)$. Thus for simple algebraic reasons (1.34) is a two-sided parametrix. $\square$

Theorem 1.15. *Let $A \in L^\mu(M, \mathbf{g})$ for $\mathbf{g} = (\gamma, \gamma - \mu, \Theta)$ be elliptic. Then $Au = f$ for $u \in H^{-\infty,\gamma}(M)$ and $f \in H^{s-\mu,\gamma-\mu}(M), s \in \mathbb{R}$, entails $u \in H^{s,\gamma}(M)$. If in addition $f \in H_S^{s-\mu,\gamma-\mu}(M)$ for an asymptotic type S associated with $(\gamma - \mu, \Theta)$ it follows that $u \in H_R^{s,\gamma}(M)$ for a resulting asymptotic type R associated with (γ, Θ).*

Proof. Let $P \in L^{-\mu}(M, \mathbf{g}^{-1})$ a parametrix of A. Then $Au = f$ implies $PAu = (1 + C)u = Pf$, i.e. $u = Pf - Cu$, and it suffices to apply (1.25). $\square$

1.3. Asymptotic parametrices in the conical case

An operator $A \in L^\mu(M, \mathbf{g})$ for $\mathbf{g} = (\gamma, \gamma - \mu, \Theta)$ is said to be flat of order $N \in \mathbb{N}$ if $\mathbf{r}^{-N} A \in L^\mu(M, \mathbf{g})$.

Proposition 1.16. *Every* $A \in L^\mu(M, \mathbf{g}), \mathbf{g} = (\gamma, \gamma - \mu, \Theta)$, *can be written in the form*

$$(1.35) \qquad A = \sum_{i=0}^{N} \mathbf{r}^i A_i \mod L_G(M, \mathbf{g})$$

for any $N \in \mathbb{N}$, *for suitable* $A_i \in L^\mu(M, \mathbf{g})$; *in particular,* $\mathbf{r}^i A_i$ *is flat of order* $i, i = 0, \dots, N$.

Proof. By definition we have A in the form (1.9). The operator $\chi A_{\mathrm{int}} \chi'$ is obviously flat of any order. Moreover, applying the Taylor formula to $h(r, z)$ at $r = 0$ we obtain a representation of $\omega r^{-\mu} \mathrm{op}_M^{\gamma - n/2}(h) \omega'$ of the desired form. So it remains to consider M which is of the form (1.22). In the case $\gamma_j = \gamma$ for all j the claimed representation is evident. For general γ_j we have at least $\gamma_0 = \gamma$. So it remains to consider the summands for $j > 0$. Let

$$M_j := r^{-\mu} \omega r^j \mathrm{op}_M^{\gamma_j - n/2}(f_j) \omega'.$$

If $\gamma_j < \gamma$ for all (or some) $j \geq 1$ we set $\tilde{M}_0 := M_0 + M_1$, and form

$$\tilde{M}_{j-1} := r^{-\mu} \omega r^j \mathrm{op}_M^{\tilde{\gamma}_j - n/2}(f_j) \omega', j \geq 2,$$

using the fact that there are weights $\tilde{\gamma}_j$ such that $\gamma - 1 \leq \tilde{\gamma}_j \leq \gamma$ with $\pi_\mathbb{C} R_j \cap \Gamma_{(n+1)/2 - \tilde{\gamma}_j} = \emptyset$. Then for $G_j := \tilde{M}_{j-1} - M_j \in L_G(M, \mathbf{g}), j \geq 2$, and for $G_1 \equiv 0$ it follows that

$$\sum_{j=0}^{k} M_j = \sum_{i=0}^{k-1} \tilde{M}_i - G_{i+1}$$

where $\mathbf{r}^{-i} \tilde{M}_i \in L_{M+G}(M, \mathbf{g})$. $\qquad \square$

Proposition 1.17. *Let* $A \in L^\mu(M, \mathbf{g}), \mathbf{g} = (\gamma, \gamma - \mu, \Theta)$, *be of the form*

$$(1.36) \qquad A = \omega r^{-\mu} \mathrm{op}_M^{\gamma - n/2}(h) \omega' + \chi A_{\mathrm{int}} \chi' + G$$

with notation as in (1.9), and let $\beta \geq 0$. *Then there is an* $A_\beta \in L^\mu(M, \mathbf{g})$ *and a* $G_\beta \in L_G(M, \mathbf{g})$ *such that* $A\mathbf{r}^\beta - \mathbf{r}^\beta A_\beta = G_\beta$.

Proof. First we have $G_\beta := G\mathbf{r}^\beta \in L_G(M, \mathbf{g})$. Moreover, $\chi A_{\mathrm{int}} \chi' \mathbf{r}^\beta = \mathbf{r}^\beta \chi A_{\mathrm{int}, \beta} \chi'$ for $A_{\mathrm{int}, \beta} := \mathbf{r}^{-\beta} \chi A_{\mathrm{int}} \mathbf{r}^\beta$ belongs to $L^\mu(M, \mathbf{g})$. Let us write A in the form (1.9). Concerning the first summand on the right of (1.36) we observe $r^\beta r^{-\beta} \mathrm{op}_M^{\gamma - n/2}(h) r^\beta$ $= r^\beta \mathrm{op}_M^{\gamma - n/2 - \beta}(T^{-\beta} h) = r^\beta \mathrm{op}_M^{\gamma - n/2}(T^{-\beta} h)$; the latter relation employs Cauchy's theorem, cf. [21, Propsition 2.3.69]. Thus $\omega r^{-\mu} \mathrm{op}_M^{\gamma - n/2}(h) \omega' \mathbf{r}^\beta = \mathbf{r}^\beta \omega r^{-\mu} \mathrm{op}_M^{\gamma - n/2}$ $(T^{-\beta} h) \omega'$, but $\omega r^{-\mu} \mathrm{op}_M^{\gamma - n/2}(T^{-\beta} h) \omega'$ belongs to $L^\mu(M, \mathbf{g})$. $\qquad \square$

Comparing (1.36) with (1.9) it remains to consider the commutation of $\mathbf{r}^\beta$ through the smoothing Mellin operator M. Clearly, close to $r = 0$ we do not make any difference between $\mathbf{r}$ and r.

Proposition 1.18. *Let* $M \in L^\mu(M, \mathbf{g}), \mathbf{g} = (\gamma, \gamma - \mu, \Theta)$, *be a smoothing Mellin operator*
(1.37)

$$M := M_0 + M_1 \ for \ M_0 := r^{-\mu}\omega \mathrm{op}_M^{\gamma-n/2}(f_0)\omega', M_1 := r^{-\mu} \sum_{j=1}^k \omega r^j \mathrm{op}_M^{\gamma_j-n/2}(f_j)\omega',$$

for Mellin symbols $f_j \in M_{R_j}^{-\infty}(X)$ *for Mellin asymptotic types* R_j, *and weights* γ_j *with* $\gamma - j \le \gamma_j \le \gamma$ *and* $\pi_\mathbb{C} R_j \cap \Gamma_{(n+1)/2-\gamma_j} = \emptyset$ *for all* j, *cf.* (1.22), *and let* $\beta \ge 0$. *Then there is an* $M_{1,\beta} \in L_{M+G}^\mu(M, \mathbf{g})$ *and a* $G_{1,\beta} \in L_G(M, \mathbf{g})$ *such that* $M_1\mathbf{r}^\beta - \mathbf{r}^\beta M_{1,\beta} = G_{1,\beta}$. *Analogously, under the condition* $\pi_\mathbb{C} R_0 \cap \Gamma_{(n+1)/2-(\gamma+\beta)} = \emptyset$ *there is an* $M_{0,\beta} \in L_{M+G}^\mu(M, \mathbf{g})$ *and a* $G_{0,\beta} \in L_G(M, \mathbf{g})$ *such that* $M_0\mathbf{r}^\beta - \mathbf{r}^\beta M_{0,\beta} = G_{0,\beta}$. *In general, for* $\beta > 0$ *and every* $0 < \varepsilon < \beta$ *we have*

(1.38) $$M_0\mathbf{r}^\beta = \mathbf{r}^{\beta-\varepsilon}\omega r^{-\mu}\mathrm{op}_M^{\gamma-n/2}(T^{-\beta+\varepsilon}f_0)\mathbf{r}^\varepsilon\omega' + G_{0,\beta-\varepsilon},$$

for a $G_{0,\beta-\varepsilon} \in L_G(M, \mathbf{g})$.

Proof. The assertion for M_0 is contained in [21, Propsition 2.3.69]. For M_1 we can argue in a similar manner. Here, because of the extra r^j-powers for $j > 0$ the condition concerning weight lines disappears. □

Let us now turn to asymptotic parametrices. Let $A \in L^\mu(M, \mathbf{g})$ be elliptic, and let $P \in L^{-\mu}(M, \mathbf{g}^{-1})$ be a parameterix, cf. Theorem 1.14. Then applying Proposition 1.16 to P we obtain an expansion

(1.39) $$P = \sum_{m=0}^N \mathbf{r}^m P_m \mod L_G(M, \mathbf{g}^{-1})$$

for every $N \in \mathbb{N}$. Let us call (1.39) an asymptotic parametrix of A. Writing A itself in the form (1.35) the operator A_0 is elliptic in $L^\mu(M, \mathbf{g})$, and $P_0 \in L^{-\mu}(M, \mathbf{g}^{-1})$ in (1.39) can be taken as a parametrix of A_0, obtained by Theorem 1.14. The construction of P_0 is easier than that of P itself because P_0 concerns the case of constant coefficients with respect to r. The idea of asymptotic paramterices is to obtain an expansion like (1.39) by an iterative process, starting with the ansatz

(1.40) $$\Big(\sum_{m=0}^N \mathbf{r}^m P_m\Big)\Big(\sum_{i=0}^N \mathbf{r}^i A_i\Big) \sim 1,$$

and, analogously,

(1.41) $$\Big(\sum_{i=0}^N \mathbf{r}^i A_i\Big)\Big(\sum_{m=0}^N \mathbf{r}^m P_m\Big) \sim 1$$

where $\sim$ indicates equality modulo a Green plus flat remainder. Let us discuss the construction of (1.40); the considerations for (1.41) are similar. The operators $\mathbf{r}^m P_m$ that we intend to obtain from successively solving (1.40) are not automatically the same as those in (1.39). Therefore, in this procedure it makes sense first to replace $\mathbf{r}^m P_m$ by $\boldsymbol{P_m}$. However, we set $\boldsymbol{P_0} := P_0$ which is the known parametrix of A_0. For $\boldsymbol{P_1}$ we define

$$(1.42) \qquad \boldsymbol{P_1} := -P_0 \mathbf{r} A_1 P_0.$$

This operator satisfies the relation

$$(1.43) \qquad \boldsymbol{P_1} A_0 + P_0 \mathbf{r} A_1 = -P_0 \mathbf{r} A_1 C_l \in L_G(M, \mathbf{g}_l)$$

where $C_l := P_0 A_0 - 1 \in L_G(M, \mathbf{g}_l)$. In general, for $j > 0$, motivated by the desirable identity $\sum_{m+i=j} \boldsymbol{P_m} \mathbf{r}^i A_i = 0$, or $\boldsymbol{P_j} A_0 = -\sum_{m+i=j, m<j} \boldsymbol{P_m} \mathbf{r}^i A_i$, we set

$$(1.44) \qquad \boldsymbol{P_j} := -\Big(\sum_{m+i=j, m<j} \boldsymbol{P_m} \mathbf{r}^i A_i \Big) P_0 \in L^{-\mu}(M, \mathbf{g}^{-1}),$$

and obtain

$$(1.45) \qquad \sum_{m+i=j} \boldsymbol{P_m} \mathbf{r}^i A_i = -\Big(\sum_{m+i=j, m<j} \boldsymbol{P_m} \mathbf{r}^i A_i \Big) C_l \in L^{-\infty}(M, \mathbf{g}_l)$$

for any j. The operators (1.44) can be expressed in terms of P_0 and $\mathbf{r}^i A_i$, for $i = 1, \ldots, j$, alone. For instance, we have

$$(1.46) \qquad \boldsymbol{P_2} = -P_0 \mathbf{r}^2 A_2 P_0 + P_0 \mathbf{r} A_1 P_0 \mathbf{r} A_1 P_0,$$

$$(1.47)$$
$$\boldsymbol{P_3} = -P_0 \mathbf{r} A_1 P_0 \mathbf{r} A_1 P_0 \mathbf{r} A_1 P_0 + P_0 \mathbf{r}^2 A_2 P_0 \mathbf{r} A_1 P_0 + P_0 \mathbf{r} A_1 P_0 \mathbf{r}^2 A_2 P_0 - P_0 \mathbf{r}^3 A_3 P_0,$$

etc. These representations allow us to establish a scheme of successively computing the asymptotics of solutions in the sense of Theorem 1.15 only by using the structure of P_0. In the applications mentioned at the beginning the operator A is a differential operator. In this case also A_m are differential operators in the cone calculus, and there are differential operators $A_{m,i}$ of analogous structure, such that

$$(1.48) \qquad A_m \mathbf{r}^i - \mathbf{r}^i A_{m,i} = 0$$

for every i. Let us assume for the moment that

$$(1.49) \qquad P_0 \mathbf{r}^i - \mathbf{r}^i P_{0,i} = G_i,$$

for every i, for remainders $G_i \in L_G(M, \mathbf{g}^{-1})$ of finite rank (which is the case when the operator M_0 contained in P_0 satisfies the corresponding condition of Proposition 1.18, namely, $\pi_{\mathbb{C}} R_0 \cap \Gamma_{(n+1)/2-(\gamma+i)} = \emptyset$). Then all $\mathbf{r}$-powers contained on the right of (1.44) can be completely commuted through the other factors, on the expense of Green remainders, and it follows that

$$(1.50) \qquad \boldsymbol{P_j} = \mathbf{r}^j P_j + D_j \text{ for suitable } P_j \in L^{-\mu}(M, \mathbf{g}^{-1})$$

and finite rank operators $D_j \in L_G(M, \mathbf{g}^{-1})$. For instance, we obtain

$$(1.51) \qquad \boldsymbol{P}_1 = -(\mathbf{r}P_{0,1} + G_1)A_1P_0 = \mathbf{r}P_1 + D_1$$

for $P_1 = -P_{0,1}A_1P_0 \in L^{-\mu}(M, \mathbf{g}^{-1})$ and $D_1 = -G_1A_1P_0 \in L_G(M, \mathbf{g}^{-1})$, etc. In this case we take the sum

$$(1.52) \qquad P_{\mathrm{as}} := \sum_{m=0}^{N} \mathbf{r}^m P_m$$

for any fixed N as an asymptotic parametrix of A. Finally, if the condition $\pi_{\mathbb{C}} R_0 \cap \Gamma_{(n+1)/2-(\gamma+i)} = \emptyset$ is not satisfied for all i, then a consequence of Proposition 1.18 is that for every $\varepsilon > 0$ we find operators of the form

$$(1.53) \qquad \boldsymbol{P}_{j,\varepsilon} = \mathbf{r}^{j-\varepsilon} P_{j,\varepsilon} + D_{j,\varepsilon} \text{ for suitable } P_{j,\varepsilon} \in L^{-\mu}(M, \mathbf{g}^{-1}), j \geq 1,$$

and finite rank operators $D_{j,\varepsilon} \in L_G(M, \mathbf{g}^{-1})$, and $\boldsymbol{P}_0 := P_0$, such that

$$(1.54) \qquad P_{\mathrm{as}} := P_0 + \sum_{m=1}^{N} \mathbf{r}^{m-\varepsilon} P_{m,\varepsilon}$$

can play the role of an asymptotic parametrix.

Theorem 1.19. *Let $A \in L^\mu(M, \mathbf{g})$ be an elliptic operator. Then for any fixed $N \in \mathbb{N}, N \geq 1$, the asymptotic parametrix P_{as} can be chosen in such a way that*

$$(1.55) \qquad P_{\mathrm{as}}A = 1 + C + F$$

for an $F \in L^\mu(M, \mathbf{g}_l)$ which is flat of order $N - 1$ and a $C \in L_G(M, \mathbf{g}_l)$.

Proof. Let us first consider the case that A is a differential operator. We already constructed P_{as}, and it remains to verify the relation (1.55). First, by construction we have

$$(1.56) \qquad \sum_{j=0}^{N} \sum_{m+i=j} \boldsymbol{P}_m \mathbf{r}^i A_i = 1 + G$$

for some $G \in L_G(M, \mathbf{g}_l)$. If P_{as} has the form $\sum_{m=0}^{N} \mathbf{r}^m P_m$, then $P_{\mathrm{as}}A$ coincides with (1.56) modulo a remainder $D + \sum_{m+i>N} \mathbf{r}^m P_m \mathbf{r}^i A_i$ for a $D \in L_G(M, \mathbf{g}_l)$. The summand $\mathbf{r}^m P_m \mathbf{r}^i A_i$ contains $m+i$ factors $\mathbf{r}$; those can be commuted through the other factors to the left (on the expense of Green remainders) which gives us at the end at least a factor $\mathbf{r}^N$ from the left. In other words, we obtain the relation (1.55). For P_{as} in the form (1.54) we can argue in a similar manner, but we may obtain on the left $\mathbf{r}^{N-\varepsilon}$ instead of $\mathbf{r}^N$. Therefore, since N is arbitrary anyway, we may be satisfied with the exponent $N - 1$ in both cases.
The construction easily extends to any elliptic (pseudo-differential) operator in the cone algebra. The only change is that the commutation relations (1.48) have to be replaced by relations of the kind $A_m \mathbf{r}^i - \mathbf{r}^{i-\varepsilon} A_{m,i,\varepsilon} = C_{i,\varepsilon}$ for any sufficiently small $\varepsilon > 0$ and a finite rank Green operator $C_{i,\varepsilon}$. $\qquad\square$

Corollary 1.20. *Theorem* 1.19 *allows us to express asymptotics of solutions to an elliptic equation* $Au = f \in H_Q^{s-\mu,\gamma-\mu}(M)$, *only by using a parametrix* P_0 *of* A_0 *and the commutation relations that are involved in the construction of* P_{as}. *From* (1.55) *it follows that* $P_{\mathrm{as}}Au = (1 + C + F)u = P_{\mathrm{as}}f \in H_S^{s,\gamma}(M)$ *for some asymptotic type S. For sufficiently large N* (*depending on the weight interval* $\Theta = (-(k+1), 0]$) *we obtain* $Fu \in H_\Theta^{s,\gamma}(M)$. *Morover,* $Cu \in H_R^{\infty,\gamma}(M)$ *for another asymptotic type R depending on C gives us finally* $u \in H_P^{s,\gamma}(M)$ *for a resulting asymptotic type P. We tacitly employed the non-asymptotic part of Theorem* 1.15, *i.e. that we already know* $u \in H^{s,\gamma}(M)$, *and the mapping properties* (1.25) *and* (1.17).

2. Elliptic edge operators

2.1. Edge-degenerate operators

We first develop some features of the edge pseudo-differential calculus including its symbolic structure which is designed to express parametrices of elliptic elements, especially parametrices of elliptic edge-degenerate differential operators, cf. (0.7). A manifold M with edge $s_1(M)$ can be represented as a quotient space $\mathbb{M}/\sim$ for the so-called stretched manifold $\mathbb{M}$ associated with M. The stretched manifold is a C^∞ manifold with boundary $\partial\mathbb{M}$ which is a locally trivial X-bundle over $s_1(M)$. The projection $\mathbb{M} \to M$ is determined by the bundle projection $\partial\mathbb{M} \to s_1(M)$ that maps the fibre X over a $y \in s_1(M)$ to y. There is a collar neighbourhood $\mathbb{V}$ of $\partial\mathbb{M}$ in $\mathbb{M}$ which can be identified with an $\overline{\mathbb{R}}_+ \times X$-bundle over $s_1(M)$. Then the pointwise projection $\overline{\mathbb{R}}_+ \times X \to X^\Delta = (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X)$ gives us a map $\mathbb{V} \to V$ where V is a neighbourhood of $s_1(M)$ in M with the structure of an X^Δ-bundle over $s_1(M)$.

On a manifold M with edge $s_1(M)$ we consider spaces of operators

$$(2.1) \qquad L^\mu(M, \mathbf{g}) \subset L_{\mathrm{cl}}^\mu(s_0(M)) \ \text{ for } \ \mathbf{g} = (\gamma, \gamma - \mu, \Theta),$$

$s_0(M) = M \setminus s_1(M)$. Modulo smoothing operators, to be specified below, the operators are locally near the edge in the splitting of variables $(r, x, y) \in X^\wedge \times \Omega, \Omega \subseteq \mathbb{R}^q$, $q = \dim s_1(M)$, of the form

$$(2.2) \qquad r^{-\mu}\mathrm{Op}_{r,y}(p) \ \text{ for a } \ p(r, y, \rho, \eta) := \tilde{p}(r, y, r\rho, r\eta),$$

$$(2.3) \qquad \tilde{p}(r, y, \tilde{\rho}, \tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, L_{\mathrm{cl}}^\mu(X; \mathbb{R}_{\tilde{\rho},\tilde{\eta}}^{1+q})).$$

Here $\mathrm{Op}_{r,y}(b)u(r, y) := \iint e^{i(r-r')\rho + i(y-y')\eta} b(r, y, r', y', \rho, \eta)u(r', y')dr'dy'đ\rho đ\eta$ for a corresponding operator-valued amplitude function b, and $đ\rho := (2\pi)^{-1}d\rho, đ\eta := (2\pi)^{-q}d\eta$ (the variable x is suppressed in most cases). In general, for $b(r, y, r', y', \rho, \eta) \in C^\infty(\mathbb{R}_+ \times \Omega \times \mathbb{R}_+ \times \Omega, L_{\mathrm{cl}}^\mu(X; \mathbb{R}^{1+q}))$ we have $\mathrm{Op}_{r,y}(b) \in L_{\mathrm{cl}}^\mu(\mathbb{R}_+ \times X \times \Omega)$, and any such operator defines a continuous map $C_0^\infty(\mathbb{R}_+ \times X \times \Omega) \to C^\infty(\mathbb{R}_+ \times X \times \Omega)$.

Let us assume for the moment that M is a compact manifold with edge (the modifications in the non-compact case will be commented afterwards). Let $U_0, \ldots, U_N$ be

an open covering of the edge $s_1(M)$ by coordinate neighbourhoods, $\{\varphi_j\}_{j=0,\dots,N}$ a subordinate partition of unity, and $\{\varphi'_j\}_{j=0,\dots,N}$ a system of functions $\varphi'_j \in C_0^\infty(U_j), \varphi'_j \succ \varphi_j$ for all j. Moreover, let $\theta \prec \theta'$ be cut-off functions on the half-axis, and choose $\psi \prec \psi'$ in $C_0^\infty(s_0(M))$ such that $\mathrm{dist}\,(\mathrm{supp}\,\psi, s_1(M))$ is small enough. Then every $A \in L_{\mathrm{cl}}^\mu(s_0(M))$ can be written in the form

$$(2.4) \qquad A = \sum_{j=0}^N \theta \varphi_j B_j \theta' \varphi'_j + \psi A_{\mathrm{int}} \psi' + C$$

with the following ingredients: $C \in L^{-\infty}(s_0(M))$, $A_{\mathrm{int}} \in L_{\mathrm{cl}}^\mu(s_0(M))$, and $B_j := \mathrm{Op}_{r,y}(b_j)$ for amplitude functions $b_j(r, y, \rho, \eta) \in C^\infty(\mathbb{R}_+ \times \Omega, L_{\mathrm{cl}}^\mu(X; \mathbb{R}_{\rho,\eta}^{1+q}))$. Here we use some convenient abbreviations. We identify U_j with an open set $\Omega \subseteq \mathbb{R}^q$ via local coordinates on the edge (for instance, we may take $\Omega = \mathbb{R}^q$), and then, via the charts $U_j \to \Omega$ we identify φ_j, φ'_j with functions in $C_0^\infty(\Omega)$. Moreover, $\mathrm{Op}_{r,y}(b_j)$ is interpreted in combination with the pull back from the local stretched wedge $X^\wedge \times \Omega$ to a corresponding open subset of $s_0(M)$.

The operators $A \in L^\mu(M, \mathbf{g})$ will be of a similar structure, namely, $C \in L^{-\infty}(M, \mathbf{g})$ (the definition will be given below, cf. the formula (2.18)), A_{int} is as in (2.4), and

$$(2.5) \qquad B_j := \mathrm{Op}_y(a_j) \text{ for an operator-valued amplitude function } a_j(y, \eta)$$

that is of the form

$$(2.6) \qquad a_j(y, \eta) := \epsilon r^{-\mu} \mathrm{Op}_r(p_j)(y, \eta)\epsilon' \bmod C^\infty(\Omega, L^{-\infty}(X^\wedge; \mathbb{R}^q))$$

for $p_j(r, y, \rho, \eta)$ as in (2.2), (2.3), moreover, $\epsilon \prec \epsilon', \theta \prec \epsilon, \theta' \prec \epsilon'$ are cut-off functions, and the remainder in (2.6) is a specific smoothing correction term coming from edge quantisation, plus contributions that are involved in the control of asymptotics (i.e. smoothing Mellin plus Green symbols).

In order to produce $a_j(y, \eta)$ we drop j for a while and apply the edge quantisation, starting from operator functions $r^{-\mu} p(r, y, \rho, \eta)$ of the form (2.2), (2.3). Here we focus on $r^{-\mu} \mathrm{Op}_r(p)(y, \eta)$ and later on pass to $\mathrm{Op}_y(r^{-\mu} \mathrm{Op}_r(p)) = r^{-\mu} \mathrm{Op}_{r,y}(p)$. Next we fix cut-off functions $\omega'' \prec \omega \prec \omega'$, and set $\chi := 1 - \omega, \chi' := 1 - \omega''$. For a function $\varphi \in C^\infty(\mathbb{R}_+)$ we write $\varphi_\eta(r) := \varphi(r[\eta])$; here $\eta \to [\eta]$ is any fixed strictly positive function in $C^\infty(\mathbb{R}^q)$ such that $[\eta] = |\eta|$ for $|\eta| \geq \varepsilon$ for some $\varepsilon > 0$. We have

$$(2.7) \quad r^{-\mu} \mathrm{Op}_r(p)(y, \eta) = r^{-\mu} \omega_\eta \mathrm{Op}_r(p)(y, \eta)\omega'_\eta + r^{-\mu} \chi_\eta \mathrm{Op}_r(p)(y, \eta)\chi'_\eta + c(y, \eta)$$

where $c(y, \eta) = r^{-\mu}\{\omega_\eta \mathrm{Op}_r(p)(y, \eta)(1 - \omega'_\eta) + (1 - \omega_\eta)\mathrm{Op}_r(p)(y, \eta)\omega''_\eta\}$ belongs to $C^\infty(\Omega, L^{-\infty}(X^\wedge; \mathbb{R}^q_\eta))$. Since $\mathrm{Op}_y(c) \in L^{-\infty}(X^\wedge \times \Omega)$ we may ignore $c(y, \eta)$ and continue with the first two summands on the right of (2.7).

Theorem 2.1. *Let $p, \tilde{p}$ be as in (2.2), (2.3); then there is an*

$$(2.8) \quad h(r, y, z, \eta) := \tilde{h}(r, y, z, r\eta) \text{ for an } \tilde{h}(r, y, z, \tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, M_{\mathcal{O}}^\mu(X, \mathbb{R}^q_{\tilde{\eta}}))$$

such that

(2.9) $\mathrm{Op}_r(p)(y,\eta) = \mathrm{op}_M^\beta(h)(y,\eta) \bmod C^\infty(\Omega, L^{-\infty}(X^\wedge; \mathbb{R}_\eta^q))$

for every $\beta \in \mathbb{R}$.

This allows us to write

(2.10) $r^{-\mu}\mathrm{Op}_r(p)(y,\eta) = r^{-\mu}\omega_\eta \mathrm{op}_M^{\gamma-n/2}(h)(y,\eta)\omega_\eta' + r^{-\mu}\chi_\eta \mathrm{Op}_r(p)(y,\eta)\chi_\eta'$

modulo another remainder in $C^\infty(\Omega, L^{-\infty}(X^\wedge; \mathbb{R}_\eta^q))$ that can be dropped again. Now let us choose cut-off functions $\epsilon'' \prec \epsilon \prec \epsilon'$. Similarly as (2.7) we can decompose once again the left hand side of (2.10) as

(2.11) $r^{-\mu}\mathrm{Op}_r(p)(y,\eta) = r^{-\mu}\epsilon \mathrm{Op}_r(p)(y,\eta)\epsilon' + r^{-\mu}(1-\epsilon)\mathrm{Op}_r(p)(y,\eta)(1-\epsilon'')$

modulo a smoothing remainder, and then

(2.12) $r^{-\mu}\mathrm{Op}_r(p)(y,\eta) = a(y,\eta) + r^{-\mu}(1-\epsilon)\mathrm{Op}_r(p)(y,\eta)(1-\epsilon'')$

modulo a smoothing remainder, where

(2.13) $a(y,\eta) = r^{-\mu}\epsilon\{\omega_\eta \mathrm{op}_M^{\gamma-n/2}(h)(y,\eta)\omega_\eta' + \chi_\eta \mathrm{Op}_r(p)(y,\eta)\chi_\eta'\}\epsilon'.$

The second summand on the right of (2.12) is supported far from $r = 0$, and it may be integrated in the interior part of (2.4). Therefore, (2.13) remains as the specific part of the edge quantisation coming from $r^{-\mu}\mathrm{Op}_r(p)(y,\eta)$.

Remark 2.2. The modification of $r^{-\mu}\mathrm{Op}_r(p)(y,\eta)$ by a smoothing operator functions close to the edge does not affect the final results concerning parametrices and asymptotics. The precise information will be obtained by adding smoothing Mellin and Green terms that automatically adjust what we lost during the edge quantisation $r^{-\mu}\mathrm{Op}_r(p)(y,\eta) \to a(y,\eta)$ which is not canonical anyway.

The main motivation of edge quantisation is the continuity of operators in weighted edge spaces. First note that when H is a Hilbert space with group action $\kappa = \{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ (i.e. a strongly continuous group of isomorphisms $\kappa_\lambda : H \to H$ with $\kappa_\lambda\kappa_\nu = \kappa_{\lambda\nu}$ for all $\lambda,\nu \in \mathbb{R}_+$) we have the space $\mathcal{W}^s(\mathbb{R}^q, H)$, defined to be the completion of $\mathcal{S}(\mathbb{R}^q, H)$ with respect to the norm $\|\langle\eta\rangle^s\kappa_{\langle\eta\rangle}^{-1}\hat{u}(\eta)\|_{L^2(\mathbb{R}^q,H)}$. Here $\hat{u}(\eta) = F_{y\to\eta}u(\eta)$ is the Fourier transform. A similar definition gives us spaces $\mathcal{W}^s(\mathbb{R}^q, E)$ for a Fréchet space E with group action. This allows us to form the spaces

(2.14) $\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\wedge))$ and $\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}_P^{s,\gamma}(X^\wedge))$

for any asymptotic type P associated with the weight data (γ, Θ). Now on a compact manifold M with edge we have the global weighted spaces

(2.15) $H^{s,\gamma}(M)$ and $H_P^{s,\gamma}(M)$

which are subspaces of $H_{\mathrm{loc}}^s(M \setminus s_1(M))$, locally near $s_1(M)$ modelled on the respective spaces (2.14).

Remark 2.3. The operator of multiplication by a function $\varphi \in C^\infty(\mathbb{M})$ induces a continuous operator

$$(2.16) \qquad \varphi : H^{s,\gamma}(M) \to H^{s,\gamma}(M) \text{ for every } s, \gamma \in \mathbb{R},$$

and the corresponding map $C^\infty(\mathbb{M}) \to \mathcal{L}(H^{s,\gamma}(M))$ is continuous. An analogous result is true of the spaces $H_P^{s,\gamma}(M)$ when the asymptotic type P satisfies the shadow condition.

The space of smoothing operators of the edge calculus

$$(2.17) \qquad L^{-\infty}(M, \mathbf{g}) \text{ for } \mathbf{g} = (\gamma, \gamma - \mu, \Theta)$$

is the set of all $C \in \bigcap_{s\in\mathbb{R}} \mathcal{L}(H^{s,\gamma}(M), H^{\infty,\gamma-\mu}(M))$ such that

$$(2.18) \qquad C : H^{s,\gamma}(M) \to H_P^{\infty,\gamma-\mu}(M), C^* : H^{s,-\gamma+\mu}(M) \to H_Q^{\infty,-\gamma}(M)$$

are continuous for every $s \in \mathbb{R}$, for certain C-dependent asymptotic types P and Q, associated with the weight data $(\gamma - \mu, \Theta)$ and $(-\gamma, \Theta)$, respectively. Here C^* is the formal adjoint of C with respect to the $H^{0,0}(M)$-scalar product. Let $L_{\text{flat}}^{-\infty}(M, \mathbf{g})$ denote the subspace of those $C \in L^{-\infty}(M, \mathbf{g})$ such that

$$(2.19) \qquad C : H^{s,\gamma}(M) \to H_\Theta^{\infty,\gamma-\mu}(M), C^* : H^{s,-\gamma+\mu}(M) \to H_\Theta^{\infty,-\gamma}(M)$$

are continuous for every $s \in \mathbb{R}$. Let us represent the stretched manifold $\mathbb{M}$ associated with M close to $\partial \mathbb{M}$ by a system of 'charts' $\mathbb{U} \to \overline{\mathbb{R}}_+ \times X \times \Omega$ (cf. also the formula (0.6)) where $\mathbb{U}$ runs over a covering of the above-mentioned set $\mathbb{V}$, $\Omega \subseteq \mathbb{R}^q$ open. Without loss of generality we assume that the transition maps $\overline{\mathbb{R}}_+ \times X \times \Omega \to \overline{\mathbb{R}}_+ \times X \times \tilde{\Omega}$ are independent of r close to $r = 0$. In other words, near $\partial \mathbb{M}$ we may fix a global axial variable $r \in \overline{\mathbb{R}}_+$. This allows us the construction of a strictly positive function $\mathbf{r}^1 \in C^\infty(\mathbb{M})$ that is equal to r for $0 \le r \le \varepsilon$ for some $\varepsilon > 0$ in the chosen representation of $\mathbb{U}$ in the variables (r, x, y). Let us form the power $\mathbf{r}^\beta$ and call $\mathbf{r}^\beta \in C^\infty(M \setminus s_1(M))$ a weight function with weight $\beta \in \mathbb{R}$. The following definition employs operator-valued symbols referring to group actions between the involved spaces. The generalities will be given in Section 2.4 below.

Definition 2.4. A $g(y, \eta) \in \bigcap_{s\in\mathbb{R}} C^\infty(U \times \mathbb{R}^q, \mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{\infty,\gamma-\mu}(X^\wedge)))$ for open $U \subseteq \mathbb{R}^p$ is said to be a Green symbol of order $\nu \in \mathbb{R}$, associated with the weight data $\mathbf{g} := (\gamma, \gamma - \mu, \Theta)$, if

$$(2.20) \qquad g(y, \eta) \in S_{\text{cl}}^\nu(U \times \mathbb{R}^q; \mathcal{K}^{s,\gamma;e}(X^\wedge), \mathcal{K}_P^{\infty,\gamma-\mu;\infty}(X^\wedge))$$

and

$$(2.21) \qquad g^*(y, \eta) \in S_{\text{cl}}^\nu(U \times \mathbb{R}^q; \mathcal{K}^{s,-\gamma+\mu;e}(X^\wedge), \mathcal{K}_Q^{\infty,-\gamma;\infty}(X^\wedge))$$

for every $s, e \in \mathbb{R}$, for certain g-dependent asymptotic types P and Q, associated with the weight data $(\gamma - \mu, \Theta)$ and $(-\gamma, \Theta)$, respectively, where g^* means the pointwise formal adjoint of g with respect to the $\mathcal{K}^{0,0}(X^\wedge)$-scalar product. Let

$$(2.22) \qquad R_G^\mu(U \times \mathbb{R}^q, \mathbf{g})_{P,Q}$$

denote the space of all such $g(y, \eta)$, and set $R_G^\mu(U \times \mathbb{R}^q, \mathbf{g}) := \bigcup_{P,Q} R_G^\mu(U \times \mathbb{R}^q, \mathbf{g})_{P,Q}$. Moreover, let $R_G^\nu(U \times \mathbb{R}^q)_\mathcal{O}$ be the subspace of those $g(y, \eta)$ such that analogues of the conditions (2.20) and (2.21) hold when the target spaces on the right are both replaced by $\mathcal{K}^{\infty,\infty;\infty}(X^\wedge)$.

Remark 2.5. For every $\varphi \in C^\infty(\overline{\mathbb{R}}_+ \times U)$ which is independent of r for large r and $g \in R_G^\mu(U \times \mathbb{R}^q, \mathbf{g})$ we have $\varphi g, g\varphi \in R_G^\mu(U \times \mathbb{R}^q, \mathbf{g})$.

Observe that $g(y, \eta) \in R_G^\nu(U \times \mathbb{R}^q)_\mathcal{O}$ is equivalent to the conditions

(2.23) $g(y, \eta), g^*(y, \eta) \in S_{\mathrm{cl}}^\nu(U \times \mathbb{R}^q; \mathcal{K}^{s,\gamma;e}(X^\wedge), \mathcal{K}^{\infty,\infty;\infty}(X^\wedge))$

for all $s, \gamma, e \in \mathbb{R}$.

Remark 2.6. An alternative to the above-mentioned edge quantisation $r^{-\mu}\mathrm{Op}_r(p)$ $(y, \eta) \to a(y, \eta)$ is to set

(2.24) $a(y, \eta) := r^{-\mu}\epsilon\mathrm{op}_M^{\gamma-n/2}(h)(y, \eta)\epsilon';$

more precisely, we have

(2.25)
$$r^{-\mu}\epsilon\{\omega_\eta\mathrm{op}_M^{\gamma-n/2}(h)(y, \eta)\omega'_\eta + \chi_\eta\mathrm{Op}_r(p)(y, \eta)\chi'_\eta\}\epsilon'$$
$$- r^{-\mu}\epsilon\mathrm{op}_M^{\gamma-n/2}(h)(y, \eta)\epsilon' \in R_G^\mu(\Omega \times \mathbb{R}^q)_\mathcal{O},$$

cf. [8].

Another typical ingredient of amplitude functions of the edge calculus are the smoothing Mellin symbols. They are defined in connection with weight data $\mathbf{g} = (\gamma, \gamma - \mu, \Theta)$ for a weight interval $\Theta := (-(k+1), 0], k \in \mathbb{N}$, and have the form

(2.26) $m(y, \eta) := r^{-\mu}\omega_\eta \sum_{j=0}^k r^j \sum_{|\alpha| \leq j} \mathrm{op}_M^{\gamma_{j\alpha}-n/2}(f_{j\alpha})(y)\eta^\alpha \omega'_\eta$

for cut-off functions ω, ω', Mellin symbols $f_{j\alpha}(y, z) \in C^\infty(\Omega, M_{R_{j\alpha}}^{-\infty}(X))$ for constant (in y) Mellin asymptotic types $R_{j\alpha}$ and weights $\gamma_{j\alpha}$ where $\gamma - j \leq \gamma_{j\alpha} \leq \gamma$ and $\pi_\mathbb{C}R_{j\alpha} \cap \Gamma_{(n+1)/2-\gamma_{j\alpha}} = \emptyset$ for all j, α.

Recall that

(2.27) $m(y, \eta) \in S_{\mathrm{cl}}^\mu(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{\infty,\gamma-\mu}(X^\wedge)),$

and

(2.28) $m(y, \eta) \in S_{\mathrm{cl}}^\mu(\Omega \times \mathbb{R}^q; \mathcal{K}_P^{s,\gamma}(X^\wedge), \mathcal{K}_Q^{\infty,\gamma-\mu}(X^\wedge)),$

for every $s \in \mathbb{R}$ and every asymptotic type P, associated with the weight data (γ, Θ) for some resulting asymptotic type Q, associated with the weight data $(\gamma - \mu, \Theta)$. Moreover, if we change the involved cut-off functions ω, ω' or the $\gamma_{j\alpha}$ (but not the Mellin symbols $f_{j\alpha}$), then we modify $m(y, \eta)$ by a Green symbol for certain asymptotic types. Let $R_{M+G}^\mu(\Omega \times \mathbb{R}^q, \mathbf{g})$ denote the space of all $m(y, \eta) + g(y, \eta)$

for arbitrary m of the form (2.26) and $g \in R_G^\mu(\Omega \times \mathbb{R}^q, \mathbf{g})$.

Finally, let

(2.29) $$R^\mu(\Omega \times \mathbb{R}^q, \mathbf{g}) \text{ for } \mathbf{g} = (\gamma, \gamma - \mu, \Theta)$$

denote the set of all operator functions of the form

(2.30) $$a(y, \eta) := a_{\mathcal{O}}(y, \eta) + (m + g)(y, \eta)$$

for $(m+g)(y,\eta) \in R_{M+G}^\mu(\Omega \times \mathbb{R}^q, \mathbf{g})$, and $a_{\mathcal{O}}(y,\eta) := r^{-\mu}\epsilon\{\omega_\eta \mathrm{op}_M^{\gamma-n/2}(h)(y,\eta)\omega_\eta' + \chi_\eta \mathrm{Op}_r(p)(y,\eta)\chi_\eta'\}\epsilon'$, or, alternatively, $a_{\mathcal{O}}(y,\eta) := r^{-\mu}\epsilon \mathrm{op}_M^{\gamma-n/2}(h)(y,\eta)\epsilon'$.
Summing up we have spaces of (y, η)-dependent operator functions on the infinite stretched cone, namely,

$$R^\mu(\Omega \times \mathbb{R}^q, \mathbf{g}) \supset R_{M+G}^\mu(\Omega \times \mathbb{R}^q, \mathbf{g}) \supset R_G^\mu(\Omega \times \mathbb{R}^q, \mathbf{g}).$$

Definition 2.7. The space

(2.31) $$L^\mu(M, \mathbf{g}) \text{ for } \mathbf{g} = (\gamma, \gamma - \mu, \Theta),$$

is defined to be the set of all operators

(2.32) $$A = \sum_{j=0}^N \theta\varphi_j \mathrm{Op}_y(a_j)\theta'\varphi_j' + \psi A_{\mathrm{int}}\psi' + C$$

for arbitrary $C \in L^{-\infty}(M, \mathbf{g})$, $A_{\mathrm{int}} \in L_{\mathrm{cl}}^\mu(s_0(M))$, $\psi, \psi' \in C_0^\infty(s_0(M))$, and $a_j(y, \eta) \in R^\mu(\Omega \times \mathbb{R}^q, \mathbf{g})$. Moreover, let $L_{M+G}^\mu(M, \mathbf{g})$ $(L_G^\mu(M, \mathbf{g}))$ denote the subspace of all elements of (2.31) such that $A_{\mathrm{int}} \equiv 0$ and $a_j \in R_{M+G}^\mu(\Omega \times \mathbb{R}^q, \mathbf{g})$ $(R_G^\mu(\Omega \times \mathbb{R}^q, \mathbf{g}))$.

For $\mathbf{g} = (\gamma, \gamma - \mu, \Theta), \Theta = (-\infty, 0]$, we set

(2.33) $$L^\mu(M, \mathbf{g}) := \bigcap_{k\in\mathbb{N}} L^\mu(M, (\gamma, \gamma - \mu, (-(k+1), 0])).$$

Analogously we define $L_{M+G}^\mu(M, \mathbf{g})$ $(L_G^\mu(M, \mathbf{g}))$ in the case $\Theta = (-\infty, 0]$ as intersections of the corresponding spaces for finite Θ.

Remark 2.8. In asymptotic considerations it may be useful to distinguish between flat operators of the edge calculus and operators encoding asymptotic properties. Let

(2.34) $$L_{\mathrm{flat}}^\mu(M, \mathbf{g}) \text{ for } \mathbf{g} = (\gamma, \gamma - \mu, \Theta),$$

$\Theta = (-(k+1), 0]$, denote the set of all $A \in L^\mu(M, \mathbf{g})$ for which in the representation (2.32) the involved local smoothing Mellin symbols belong to $C^\infty(\Omega, M_{\mathcal{O}}^{-\infty}(X))$, the local Green symbols to $R_G^\mu(\Omega \times \mathbb{R}^q)_{\Theta,\Theta}$, and $C \in L_{\mathrm{flat}}^{-\infty}(M, \mathbf{g})$. Then we have

(2.35) $$L^\mu(M, \mathbf{g}) = L_{\mathrm{flat}}^\mu(M, \mathbf{g}) + L_{M+G}^\mu(M, \mathbf{g}).$$

Spaces of flat operators for $\Theta = (-\infty, 0]$ are defined again as intersections of corresponding spaces for finite Θ; in particular, the local Green symbols then belong to $R_G^\mu(\Omega \times \mathbb{R}^q)_{\mathcal{O}}$, and we have a relation like (2.35) also in the case of infinite Θ. There is an immediate generalisation of Definition 2.7 to the case of

a non-compact manifold M with edge. The smoothing operators are based on mapping properties of the kind $H^{s,\gamma}_{[\mathrm{comp})}(M) \to H^{\infty,\gamma-\mu}_{[\mathrm{loc}),P}(M)$ and $H^{s,-\gamma+\mu}_{[\mathrm{comp})}(M) \to H^{\infty,-\gamma}_{[\mathrm{loc}),Q}(M)$, respectively, while the first sum on the right of (2.32) is to be replaced by a corresponding locally finite sum referring to an infinite covering $\{U_j\}_{j\in\mathbb{N}}$ of the edge.

Remark 2.9. The space $\mathrm{Diff}^\mu_{\mathrm{deg}}(M)$ belongs to (2.33) for every weight $\gamma \in \mathbb{R}$.

Remark 2.10. We have $L^\mu(M,\mathbf{g}) \cap L^{-\infty}(s_0(M)) = L^\mu_{M+G}(M,\mathbf{g})$.

The operator spaces of Definition 2.7 have some natural properties in connection with the above-mentioned weight functions. In the following consideration we assume, for simplicity, $\Theta := (-\infty, 0]$. Let us call an $A \in L^\mu(M,\mathbf{g})$ for $\mathbf{g} = (\gamma, \gamma - \mu, \Theta)$, flat of order $N \in \mathbb{N}$ if $\mathbf{r}^{-N} A \in L^\mu(M,\mathbf{g})$.
Observe that the operators $\psi A_{\mathrm{int}} \psi'$ in (2.32) are flat of infinite order (i.e. for arbitrary N).

Proposition 2.11. *Every $A \in L^\mu(M,\mathbf{g})$ can be written in the form*

$$(2.36) \qquad\qquad A = \sum_{l=0}^{N} \mathbf{r}^l A_l \ \mathrm{mod}\, L^\mu_G(M,\mathbf{g})$$

for any $N \in \mathbb{N}$, for suitable $A_l \in L^\mu(M,\mathbf{g})$ where A_l is flat of order l for every l.

Proof. By virtue of Definition 2.7 it suffices to write the local amplitude functions a_j in (2.32) in the form $a_j = \sum_{l=0}^{N} r^l a_{jl} + a_{j,(N)}$ modulo $R^\mu_G(\Omega \times \mathbb{R}^q, \mathbf{g})$, for elements $a_{jl}, a_{j,(N)} \in R^\mu(\Omega \times \mathbb{R}^q, \mathbf{g})$ such that $r^{-N} a_{j,(N)} \in R^\mu(\Omega \times \mathbb{R}^q, \mathbf{g})$. For any fixed j we now drop the subscript and write $a_j(y,\eta) =: a(y,\eta)$ in the form $a(y,\eta) := a_{\mathcal{O}}(y,\eta) + m(y,\eta)$ modulo $R^\mu_G(\Omega \times \mathbb{R}^q, \mathbf{g})$, cf. (2.30), where $a_{\mathcal{O}}(y,\eta) := r^{-\mu} \epsilon \mathrm{op}^{\gamma-n/2}_M(h)(y,\eta) \epsilon'$ for an $h(r,y,z,\eta) = \tilde{h}(r,y,z,r\eta), \tilde{h}(r,y,z,\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, M_{\mathcal{O}}(X; \mathbb{R}^q_{\tilde{\eta}}))$, and $m(y,\eta)$ as in (2.26). What concerns $a_{\mathcal{O}}(y,\eta)$ we simply apply Taylor's formula to $\tilde{h}(r,y,z,\tilde{\eta})$ at $r = 0$. For $m(y,\eta)$ the desired representation is immediate from (2.26) when $\gamma_{j\alpha} = \gamma$ for all j, α, namely, $m(y,\eta) = \sum_{l=0}^{N} r^l m_l(y,\eta)$, say, for $N \geq k$, with

$$m_0(y,\eta) := r^{-\mu} \omega_\eta \sum_{j=0}^{k} r^j \sum_{|\alpha|=j} \mathrm{op}^{\gamma_{j\alpha}-n/2}_M(f_{j\alpha})(y)\eta^\alpha \omega'_\eta, \dots,$$

$$m_l(y,\eta) := r^{-\mu} \omega_\eta \sum_{j=l}^{k} r^{j-l} \sum_{|\alpha|=j-l} \mathrm{op}^{\gamma_{j\alpha}-n/2}_M(f_{j\alpha})(y)\eta^\alpha \omega'_\eta.$$

In the general case we can write $m(y,\eta) = \sum_{l=0}^{N} r^l \tilde{m}_l(y,\eta)$, for

$$\tilde{m}_0(y,\eta) := m_0(y,\eta) + r m_1(y,\eta),$$

$$\tilde{m}_l(y,\eta) := r^{-\mu+1}\omega_\eta \sum_{j=l+1}^{k} r^{j-(l+1)} \sum_{|\alpha|=j-(l+1)} \mathrm{op}_M^{\tilde{\gamma}_{j\alpha}-n/2}(f_{j\alpha})(y)\eta^\alpha \omega_\eta',$$

for $l = 1, 2, \ldots$, for weights $\tilde{\gamma}_{j\alpha}$ satisfying the conditions $\gamma - 1 \leq \tilde{\gamma}_{j\alpha} \leq \gamma$ and $\pi_{\mathbb{C}}R_{j\alpha} \cap \Gamma_{(n+1)/2-\tilde{\gamma}_{j\alpha}} = \emptyset$ for all $|\alpha| = j - (l+1)$. Here we use the fact that the change of weights which is always possible in the indicated way, gives rise to admitted Green remainders. $\qquad\square$

Example 1. Let us illustrate Proposition 2.11 for $A \in \mathrm{Diff}^\mu_{\mathrm{deg}}(M)$, cf. Remark 2.9. Any such A has the form $A = A_{\mathrm{mel}} + A_{\mathrm{flat}}$ where $A_{\mathrm{flat}} := \psi A_{\mathrm{int}}\psi'$ in the notation of (2.32), $A_{\mathrm{int}} \in \mathrm{Diff}^\mu(s_0(M))$, and A_{mel} is locally near $s_1(M)$ in the splitting of variables $(r, x, y) \in \mathbb{R}_+ \times X \times \Omega$ a Mellin operator $A_{\mathrm{mel}} = \epsilon(r)r^{-\mu}\mathrm{Op}_y\mathrm{op}_M^{\gamma-n/2}(h)$ for an $h(r,y,z,\eta) = \tilde{h}(r,y,z,r\eta)$, $\tilde{h}(r,y,z,\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, M_{\mathcal{O}}^\mu(X;\mathbb{R}^q_{\tilde{\eta}}))$ (here taking values in parameter-dependent differential operators on X), and a cut-off function ϵ. As noted in the proof of Proposition 2.11 the l-th term of (2.36) is locally near $r = 0$ of the form $\epsilon(r)r^{-\mu+l}\mathrm{Op}_y\mathrm{op}_M^{\gamma-n/2}(h_l)$ for $h_l(r,y,z,\eta) := \tilde{h}_l(y,z,r\eta)$, determined by the Taylor expansion $\tilde{h}(r,y,z,\tilde{\eta}) = \sum_{l\in\mathbb{N}} r^l \tilde{h}_l(y,z,\tilde{\eta})$ at $r = 0$.

Proposition 2.12. $A \in L^\mu(M,\mathbf{g})$ and $\varphi \in C^\infty(\mathbb{M})$ entails $\varphi A, A\varphi \in L^\mu(M,\mathbf{g})$.

Proof. From Remark 2.3 we see immediately that $L^{-\infty}(M,\mathbf{g})$ is preserved by the multiplication by φ. Moreover, the operators $\psi A_{\mathrm{int}}\psi'$ in (2.32) obviously survive such multiplications. The amplitude functions $a_j \in R^\mu(\Omega \times \mathbb{R}^q, \mathbf{g})$ can be multiplied from the left by φ which is trivial for $a_{\mathcal{O}}$. Moreover, Theorem 2.19 below on the level of edge amplitide functions shows that $a_{\mathcal{O}}$ also admits the multiplication from the right by φ. The smoothing Mellin + Green symbols can be multilied from both sides by φ. $\qquad\square$

2.2. Ellipticity in the edge algebra

The ellipticity of an operator $A \in L^\mu(M,\mathbf{g})$, $\mathbf{g} = (\gamma, \gamma - \mu, \Theta)$, relies on the principal symbolic structure $\sigma(A) = (\sigma_0(A), \sigma_1(A))$, generalising the one of $A \in \mathrm{Diff}^\mu_{\mathrm{deg}}(M)$, mentioned at the very beginning. The interior symbol $\sigma_0(A)$ is nothing else than the homogeneous principal symbol of A as an operator in $L^\mu_{\mathrm{cl}}(s_0(M))$. Moreover, locally close to $s_1(M)$ the operator A is equal to (2.2) for (2.3), modulo a smoothing operator. Then, $\tilde{\sigma}_0(A) := \tilde{p}_{(\mu)}(r,y,\tilde{\rho},\tilde{\eta})$, the parameter-dependent principal symbol of (2.3), is the pseudo-differential analogue of (0.9), also called the rescaled symbol of A. Moreover, $\sigma_1(A)$, the homogeneous principal edge symbol of the operator A, is the pseudo-differential generalisation of (0.8), defined as

$$\sigma_1(A)(y,\eta) := r^{-\mu}\{\omega_{|\eta|}\mathrm{op}_M^{\gamma-n/2}(h_0)(y,\eta)\omega_{|\eta|}' + \chi_{|\eta|}\mathrm{Op}_r(p_0)(y,\eta)\chi_{|\eta|}'\}$$

(2.37)

$$+ r^{-\mu}\omega_{|\eta|} \sum_{j=0}^{k} r^j \sum_{|\alpha|=j} \mathrm{op}_M^{\gamma_{j\alpha}-n/2}(f_{j\alpha})(y)\eta^\alpha \omega_{|\eta|}' + g_{(\mu)}(y,\eta),$$

$(y, \eta) \in T^*(s_1(M)) \setminus 0$, $\omega_{|\eta|}(r) := \omega(r|\eta|)$, etc. Here

(2.38) $p_0(r, y, \rho, \eta) := \tilde{p}(0, y, r\rho, r\eta)$, $h_0(r, y, z, \eta) := \tilde{h}(0, y, z, r\eta)$.

Moreover, $m + g$ is the smoothing Mellin plus Green part of the local amplitude function of A, cf. (2.30), where m is written in the form (2.26), while $g_{(\mu)}$ is the (twisted homogeneous) principal symbol of the Green symbol g. The principal edge symbol defines a family of continuous operators

(2.39) $\sigma_1(A)(y, \eta) : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\mu, \gamma-\mu}(X^\wedge)$,

smoothly depending on (y, η) and homogeneous in the sense

(2.40) $\sigma_1(A)(y, \lambda\eta) = \lambda^\mu \kappa_\lambda \sigma_1(A)(y, \eta) \kappa_\lambda^{-1}$

for all $\lambda \in \mathbb{R}_+, (y, \eta) \in T^*(s_1(M)) \setminus 0$.

Definition 2.13. Let $A \in L^\mu(M, \mathbf{g}), \mathbf{g} = (\gamma, \gamma - \mu, \Theta)$.

(i) The operator A is called σ_0-elliptic, if $\sigma_0(A)$ never vanishes on $T^*(s_0(M)) \setminus 0$, and if in addition close to the edge $\tilde{\sigma}_0(A)(r, x, y, \rho, \xi, \eta) \neq 0$ for $(\rho, \xi, \eta) \neq 0$, up to $r = 0$.
(ii) The operator A is called σ_1-elliptic if (2.40) is a family of isomorphisms for all $(y, \eta) \in T^*(s_1(M)) \setminus 0$ for any $s \in \mathbb{R}$.

We call A elliptic if it satisfies the conditions (i), (ii).

Remark 2.14. The definition of ellipticity admits several generalisations, cf. [18], namely, in form of bijective families of 2×2 block matrices

(2.41) $\sigma_1(\mathcal{A})(y, \eta) : \mathcal{K}^{s,\gamma}(X^\wedge) \oplus J_{-,y} \to \mathcal{K}^{s-\mu, \gamma-\mu}(X^\wedge) \oplus J_{+,y}, \sigma_1(A) = \sigma_1(\mathcal{A})_{11}$,

where $J_{\pm,y}$ are the fibres of complex vector bundles $J_\pm$ over $s_1(M)$. If we write $\sigma_1(\mathcal{A}) = (\sigma_1(\mathcal{A}_{ij}))_{i,j=1,2}$, then $\sigma_1(\mathcal{A}_{21})$ is the symbol of an elliptic edge trace condition and $\sigma_1(\mathcal{A}_{12})$ of an elliptic edge potential condition, while $\sigma_1(\mathcal{A}_{22})$ is the principal symbol of a classical pseudo-differential operator over $s_1(M)$ (operating between sections of $J_\pm$). The bijectivity of (2.14) is an analogue of the Shapiro-Lopatinskij condition, known from the ellipticity of (pseudo-differential) boundary value problems, cf. [2], or [14]. In our applications we focus on the case when $J_\pm$ are of fibre-dimension 0 which corresponds to the bijectivity of (2.39). In general the bundles $J_\pm$ may depend on the weight γ. It is an interesting aspect of ellipticity in the edge case to identify those weights where the extra conditions are not necessary. Another generalisation concerns the case when for topological reasons the family (2.39) does not admit an extension to a bijective block matrix family (2.41). Such a situation is well-known in analogous form in boundary value problems, cf. [1]. A block matrix pseudo-differential algebra that extends the one of [2] which covers all these situations, including the parametrices in the elliptic case, has been introduced in [22]; the edge analogue in a similar framework has been studied in [23].

Before we start explaining the structure of parametrices of elliptic edge operators we complete the material on the operator spaces of Definition 2.7.

Let us consider $L^\mu(M, \mathbf{g})$ for $\mathbf{g} = (\gamma, \gamma - \mu, \Theta)$ and set

$$(2.42) \qquad \sigma\big(L^\mu(M, \mathbf{g})\big) := \{\sigma(A) = (\sigma_0(A), \sigma_1(A)) : A \in L^\mu(M, \mathbf{g})\}$$

which is a vector space. The components of $\sigma(A)$ are uniquely determined by A, and hence we have a linear map

$$(2.43) \qquad \sigma : L^\mu(M, \mathbf{g}) \to \sigma\big(L^\mu(M, \mathbf{g})\big).$$

Proposition 2.15. *The principal symbolic map* (2.43) *has a right inverse*

$$(2.44) \qquad \mathrm{op} : \sigma\big(L^\mu(M, \mathbf{g})\big) \to L^\mu(M, \mathbf{g}),$$

i.e. $\sigma \circ \mathrm{op} = \mathrm{id}$ *on the space* $\sigma\big(L^\mu(M, \mathbf{g})\big)$.

Proof. The construction of op is local in nature. In particular we may argue for symbols or operators over an open set $V \subset M$ separately for $\mathrm{dist}\,(V, s_0(M)) > 0$ and $V \cap s_0(M) = \emptyset$. The assertion in the first case is well-known; here only σ_0 is involved. Therefore, we concentrate on a neighbourhood close to $s_1(M)$, represented by $X^\Delta \times \Omega$, in the splitting of variables $(r, x, y) \in \Sigma^\wedge \times \Omega$ where $\Sigma \subseteq \mathbb{R}^n$ corresponds to a chart $U \to \Sigma$ on X. The first symbolic component is a function of the form $r^{-\mu} p_{(\mu)}(r, x, y, \rho, \xi, \eta)$ with $p_{(\mu)}(r, x, y, \rho, \xi, \eta) = \tilde{p}_{(\mu)}(r, x, y, r\rho, \xi, r\eta)$ for a $\tilde{p}_{(\mu)}(r, x, y, \tilde{\rho}, \xi, \tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Sigma \times \Omega \times (\mathbb{R}^{1+n+q}_{\tilde{\rho}, \xi, \tilde{\eta}} \setminus \{0\}))$. Without loss of generality we may assume that $p_{(\mu)}(r, x, y, \rho, \xi, \eta)$ vanishes with respect to x off some compact subset of Σ; the general case can be treated by gluing together the constructions for an open covering of X by coodinate neighbourhoods, using a subordinate partition of unity. For a similar reason it suffices to assume $p_{(\mu)}(r, x, y, \rho, \xi, \eta) \equiv 0$ for $r > \varepsilon$ for some small $\varepsilon > 0$. Let (a_0, a_1) be the given element of $\sigma\big(L^\mu(M, \mathbf{g})\big)$, namely,

$$(2.45) \qquad a_0 = r^{-\mu} p_{(\mu)}(r, x, y, \rho, \xi, \eta),$$

and

$$(2.46) \qquad a_1 = r^{-\mu}\{\omega_{|\eta|}\mathrm{op}_M^{\gamma - n/2}(h_0')(y, \eta)\omega_{|\eta|}' + \chi_{|\eta|}\mathrm{Op}_r(p_0')(y, \eta)\chi_{|\eta|}'\} + c_1$$

for

$$(2.47) \qquad c_1 := r^{-\mu}\omega_{|\eta|}\sum_{j=0}^{k} r^j \sum_{|\alpha|=j} \mathrm{op}_M^{\gamma_{j\alpha} - n/2}(f_{j\alpha})(y)\eta^\alpha \omega_{|\eta|}' + g_{(\mu)}(y, \eta),$$

for certain $p_0'(r, y, \rho, \eta) = \tilde{p}_0'(y, r\rho, r\eta), \tilde{p}_0'(y, \tilde{\rho}, \tilde{\eta}) \in C^\infty(\Omega, L^\mu_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\tilde{\rho}, \tilde{\eta}}))$ and $h_0'(r, y, z, \eta) = \tilde{h}_0'(y, z, r\eta), \tilde{h}_0'(y, z, \tilde{\eta}) \in C^\infty(\Omega, M^\mu_\mathcal{O}(X; \mathbb{R}^q_{\tilde{\eta}}))$ where $\mathrm{Op}_r(p_0')(y, \eta) = \mathrm{op}_M^\beta(h_0')(y, \eta) \bmod C^\infty(\Omega, L^{-\infty}(X^\wedge; \mathbb{R}^q_\eta))$. The compatibility condition between a_0 and a_1 consists of the fact that the parameter-dependent principal symbol $\tilde{p}_{0,(\mu)}'(x, y, \tilde{\rho}, \xi, \tilde{\eta})$ of $\tilde{p}_0'(y, \tilde{\rho}, \tilde{\eta})$ satisfies the relation

$$(2.48) \qquad p_{(\mu)}(0, x, y, \rho, \xi, \eta) = \tilde{p}_{0,(\mu)}'(x, y, r\rho, \xi, r\eta).$$

Now a staightforward modification of a well-known extension theorem of Seeley tells us that there is a $\tilde{p}(r, x, y, \tilde{\rho}, \xi, \tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, L^\mu_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\tilde{\rho}, \tilde{\eta}}))$ with the given $\tilde{p}_{(\mu)}(r, x, y, \tilde{\rho}, \xi, \tilde{\eta})$ as its parameter-dependent homogeneous principal symbol, and

$\tilde{p}(0, y, \tilde{\rho}, \tilde{\eta}) = \tilde{p}'_0(y, \tilde{\rho}, \tilde{\eta})$. By virtue of Theorem 2.1 we can choose an $\tilde{h}(r, y, z, \tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, M^\mu_{\mathcal{O}}(X, \mathbb{R}^q_{\tilde{\eta}}))$ such that $h(r, y, z, \eta) := \tilde{h}(r, y, z, r\eta)$ satisfies the relation $\mathrm{Op}_r(p)(y, \eta) = \mathrm{op}^\beta_M(h)(y, \eta) \bmod C^\infty(\Omega, L^{-\infty}(X^\wedge; \mathbb{R}^q_\eta))$ for every $\beta \in \mathbb{R}$. It follows that $\tilde{h}(0, y, z, \tilde{\eta}) - \tilde{h}'_0(y, z, \tilde{\eta}) \in C^\infty(\Omega, M^{-\infty}_{\mathcal{O}}(X, \mathbb{R}^q_{\tilde{\eta}}))$. Finally with the constructed data we form the operator-valued amplitude function (2.13) and set $B := \mathrm{Op}_y(a)$. Then $\sigma_0(B) = a_0$ and

$$(2.49) \qquad \sigma_1(B) = r^{-\mu}\omega_{|\eta|}\mathrm{op}^{\gamma - n/2}_M(h_0)(y, \eta)\omega'_{|\eta|} + r^{-\mu}\chi_{|\eta|}\mathrm{Op}_r(p_0)(y, \eta)\chi'_{|\eta|}$$

for $h_0(r, y, z, \eta) = \tilde{h}(0, y, z, r\eta), p_0(r, y, \rho, \eta) = \tilde{p}(0, y, r\rho, r\eta)$. For an excision function $\vartheta(\eta)$ in $\mathbb{R}^q$ we form

$$(2.50) \qquad C := \mathrm{Op}_y(\vartheta(\eta)c_1(y, \eta)) + r^{-\mu}\mathrm{Op}_y(\omega_\eta \mathrm{op}^{\gamma - n/2}_M(h'_0 - h_0)(y, \eta)\omega'_\eta).$$

Then for $A := B + C \in L^\mu(B, \mathbf{g})$ we obtain $\sigma_0(A) = a_0, \sigma_1(A) = a_1$, and hence we may set $A := \mathrm{op}(a_0, a_1)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Any choice of (2.44) is called an operator convention of the edge calculus.

The operators $A \in L^{\mu-1}(M, \mathbf{g}) \subset L^{\mu-1}_{\mathrm{cl}}(s_0(M))$ have again a homogeneous principal symbol $\sigma^{\mu-1}_0(A)$ of order $\mu - 1$ in the standard sense, and a homogeneous principal edge symbol $\sigma^{\mu-1}_1(A)$ of order $\mu - 1$,

$$(2.51) \qquad\qquad \sigma^{\mu-1}_1(A)(y, \eta) : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\mu+1, \gamma - \mu}(X^\wedge),$$

parametrised by $(y, \eta) \in T^*(s_1(M) \setminus 0)$, (twisted) homogeneous in the sense $\sigma^{\mu-1}_1(A)(y, \lambda\eta) = \lambda^{\mu-1}\kappa^{-1}_\lambda \sigma^{\mu-1}_1(A)(y, \eta)\kappa_\lambda, \lambda \in \mathbb{R}_+$. Successively we now define the operator spaces $L^{\mu-j}(M, \mathbf{g})$ for $\mathbf{g} = (\gamma, \gamma - \mu, \Theta), j = 1, 2, \ldots$, by the condition of vanishing $\sigma^{\mu-(j-1)}(A) = (\sigma^{\mu-(j-1)}_0(A), \sigma^{\mu-(j-1)}_1(A))$. Let $L^{\mu-j}_{M+G}(M, \mathbf{g}) := L^\mu_{M+G}(M, \mathbf{g}) \cap L^{\mu-j}(M, \mathbf{g}), L^{\mu-j}_G(M, \mathbf{g}) := L^\mu_G(M, \mathbf{g}) \cap L^{\mu-j}(M, \mathbf{g})$ for any $j \in \mathbb{N}$.

Theorem 2.16. *Let $A_j \in L^{\mu-j}(M, \mathbf{g}), j \in \mathbb{N}, \Theta = (-(k+1), 0]$, be an arbitrary sequence, and assume that the asymptotic types involved in the Green operators are independent of j. Then there is an $A \in L^\mu(M, \mathbf{g})$ such that $A - \sum^N_{j=0} A_j \in L^{\mu-(N+1)}(M, \mathbf{g})$ for every $N \in \mathbb{N}$, and A is unique modulo $L^{-\infty}(M, \mathbf{g})$.*

Similarly as in the standard pseudo-differential calculus A is called an asymptotic sum of the $A_j, j \in \mathbb{N}$, written $A \sim \sum^\infty_{j=0} A_j$.

Remark 2.17. An operator $A \in L^m(M, \mathbf{g})$ for $\mathbf{g} = (\gamma, \gamma - \mu, \Theta), \mu - m \in \mathbb{N}$, induces continuous operators

$$(2.52) \qquad A : H^{s,\gamma}(M) \to H^{s-m, \gamma - \mu}(M), H^{s,\gamma}_P(M) \to H^{s-m, \gamma - \mu}_Q(M)$$

for every $s \in \mathbb{R}$ and any asymptotic type P associated with (γ, Θ), for some resulting Q associated with $(\gamma - \mu, \Theta)$.

Theorem 2.18. *The composition of operators induces a bilinear map*

(2.53) $$L^m(M, \mathbf{g}) \times L^n(M, \mathbf{h}) \to L^{m+n}(M, \mathbf{g} \circ \mathbf{h})$$

for $\mu - m, \nu - n \in \mathbb{N}$, and $\mathbf{g} = (\gamma - \nu, \gamma - (\mu + \nu), \Theta), \mathbf{h} = (\gamma, \gamma - \nu, \Theta), \mathbf{g} \circ \mathbf{h} = (\gamma, \gamma - (\mu + \nu), \Theta)$, and we have

(2.54) $$\sigma^{m+n}(AB) = \sigma^m(A)\sigma^n(B)$$

with componentwise multiplication. Moreover, if one of the factors belongs to the space with subscript $M + G$ (G), then the same is true of the composition.

Theorem 2.19. *Let $A \in L^m(M, \mathbf{g})$ for $\mathbf{g} = (\gamma, \gamma - \mu, \Theta), \mu - m \in \mathbb{N}$; then for the formal adjoint we have $A^* \in L^m(M, \mathbf{g}^*)$ for $\mathbf{g}^* = (-\gamma + \mu, -\gamma, \Theta)$ (defined by $(Au, v)_{H^{0,0}(M)} = (u, A^*v)_{H^{0,0}(M)}$ for all $u, v \in C_0^\infty(s_0(M)))$, and*

(2.55) $$\sigma^m(A^*) = \sigma^m(A)^*$$

componentwise, i.e. $\sigma_0^m(A^*) = \overline{\sigma_0^m(A)}$, and $\sigma_1^m(A^*) = \sigma_1^m(A)^*$ where '$*$' on the right means the (y, η)-wise formal adjoint of $\sigma_1^m(A) : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-m,\gamma-\mu}(X^\wedge)$ in the sense $(\sigma_1^m(A)f, g)_{\mathcal{K}^{0,0}(X^\wedge)} = (f, \sigma_1^m(A)^*g)_{\mathcal{K}^{0,0}(X^\wedge)}$ for $f, g \in C_0^\infty(X^\wedge)$. Moreover, if A belongs to the space with subscript $M + G$ (G), then the same is true of the formal adjoint.

2.3. Asymptotic parametrices in the edge case

We now turn to the construction of parametrices of elliptic operators. The new aspect here are asymptotic parametrices, i.e. with flat and Green operators as left over terms. Those give rise to a transparent description of asymptotics to elliptic operators in the edge calculus. We first outline the known general ideas of the parametrix construction. Let $A \in L^\mu(M, \mathbf{g})$, $\mathbf{g} = (\gamma, \gamma - \mu, \Theta)$. The first step is to employ σ_0-ellipticity of A, cf. Definition 2.13. Apart from the standard ellipticity in $L_{\mathrm{cl}}^\mu(s_0(M))$ the operator A is locally near $s_1(M)$ of the form $A = r^{-\mu}\mathrm{Op}_{r,y}(p)$ modulo a smoothing operator, where p is as in (2.2), (2.3), and $\tilde{p}(r, y, \tilde{\rho}, \tilde{\eta})$ takes values in parameter-dependent elliptic operators in $L_{\mathrm{cl}}^\mu(X; \mathbb{R}_{\tilde{\rho},\tilde{\eta}}^{1+q})$ up to $r = 0$.

Lemma 2.20. *Under the above-mentioned conditions on p there exists a $p^{(-1)}(r, y, \rho, \eta) = \tilde{p}^{(-1)}(r, y, r\rho, r\eta)$, $\tilde{p}^{(-1)}(r, y, \tilde{\rho}, \tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+, L_{\mathrm{cl}}^{-\mu}(X; \mathbb{R}_{\tilde{\rho},\tilde{\eta}}^{1+q}))$, such that*

(2.56) $$\begin{aligned} r^\mu p^{(-1)}(r, y, \rho, \eta) \, \sharp_{r,y} \, r^{-\mu}p(r, y, \rho, \eta) &\sim 1, \\ r^{-\mu}p(r, y, \rho, \eta) \, \sharp_{r,y} \, r^\mu p^{(-1)}(r, y, \rho, \eta) &\sim 1, \end{aligned}$$

and equivalence means equality of corresponding asymptotic sums modulo smoothing families. Here, $\sharp_{r,y}$ between operator families $a(r, y, \rho, \eta), b(r, y, \rho, \eta)$ means the Leibniz product in the respective variables, i.e. the asymptotic sum

$$\sum_{\alpha \in \mathbb{N}^{1+q}} 1/\alpha! \, \partial_{\rho,\eta}^\alpha a(r, y, \rho, \eta)(D_{r,y}^\alpha b(r, y, \rho, \eta)).$$

Let us first assume that our elliptic operator has the form

$$(2.57) \qquad A = \sum_{j=0}^{N} \vartheta_j \mathrm{Op}_y(a_j)\vartheta_j' + \psi A_{\mathrm{int}}\psi'$$

for $\vartheta_j := \theta \varphi_j \prec \vartheta_j' := \theta' \varphi_j'$, and without loss of generality, $\sum_{j=0}^{N} \vartheta_j + \psi \equiv 1, \psi \prec \psi'$, and $A_{\mathrm{int}} = A$ modulo $L^{-\infty}(s_0(M))$, as an operator in $L_{\mathrm{cl}}^{\mu}(s_0(M))$. Moreover, $a_j(y,\eta)$ is given by $r^{-\mu}\epsilon\{\omega_\eta \mathrm{op}_M^{\gamma-n/2}(h_j)(y,\eta)\omega_\eta' + \chi_\eta \mathrm{Op}_r(p_j)(y,\eta)\chi_\eta'\}\epsilon'$, or, alternatively, $r^{-\mu}\epsilon \mathrm{op}_M^{\gamma-n/2}(h_j)(y,\eta)\epsilon'$, i.e., compared with (2.32), without smoothing Mellin and Green terms, and we assume $C = 0$. For instance, every $A \in \mathrm{Diff}_{\mathrm{deg}}^{\mu}(M)$ can be written in this way. Clearly $p_j(r,y,\rho,\eta) = \tilde{p}_j(r,y,r\rho,r\eta)$ refers to the corresponding local representation of A over $X^\wedge \times \Omega$, and $h_j(r,y,z,\eta) = \tilde{h}_j(r,y,z,r\eta)$ is associated with p_j as in Theorem 2.1. Let us now fix j and drop it again. Applying Theorem 2.1 with $p^{(-1)}(r,y,\rho,\eta)$ of Lemma 2.20 we associate an $h^{(-1)}(r, y,z,\eta) = \tilde{h}^{(-1)}(r,y,z,r\eta), \tilde{h}^{(-1)}(r,y,z,\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, M_{\mathcal{O}}^{-\mu}(X,\mathbb{R}_{\tilde{\eta}}^q))$. This can be done for every j, and we set

$$(2.58) \quad a_j^{-1}(y,\eta) := r^{\mu}\epsilon\{\omega_\eta \mathrm{op}_M^{\gamma-\mu-n/2}(h_j^{(-1)})(y,\eta)\omega_\eta' + \chi_\eta \mathrm{Op}_r(p_j^{(-1)})(y,\eta)\chi_\eta'\}\epsilon',$$

or, alternatively, $a_j^{-1}(y,\eta) := r^{\mu}\epsilon \mathrm{op}_M^{\gamma-n/2}(h_j^{(-1)})(y,\eta)\epsilon'$. Moreover, let $A_{\mathrm{int}}^{(-1)} \in L_{\mathrm{cl}}^{-\mu}(s_0(M))$ be any parametrix of the elliptic operator A_{int}.

Lemma 2.21. *The operator*

$$(2.59) \qquad A^{(-1)} := \sum_{j=0}^{N} \vartheta_j \mathrm{Op}_y(a_j^{-1})\vartheta_j' + \psi A_{\mathrm{int}}^{(-1)}\psi'$$

has the properties $A^{(-1)} \in L^{-\mu}(M,\mathbf{g}^{-1})$ *for* $\mathbf{g}^{-1} = (\gamma - \mu, \gamma, \Theta)$, *and*

$$(2.60) \qquad A^{(-1)}A = 1 \bmod L_{M+G}^0(M,\mathbf{g}_l), \quad AA^{(-1)} = 1 \bmod L_{M+G}^0(M,\mathbf{g}_r)$$

for $\mathbf{g}_l = (\gamma, \gamma, \Theta), \mathbf{g}_r = (\gamma - \mu, \gamma - \mu, \Theta)$.

Proof. By virtue of the definition of $A_{\mathrm{int}}^{(-1)}$ and the construction of the sum on the right of (2.59) we have $A_{\mathrm{int}}^{(-1)} = A^{(-1)}$ modulo $L^{-\infty}(s_0(M))$, as an operator in $L_{\mathrm{cl}}^{-\mu}(s_0(M))$. It follows that $A^{(-1)}$ is a parametrix of A over $s_0(M)$ in the standard sense. Since both $A : H^{s,\gamma}(M) \to H^{s-\mu,\gamma-\mu}(M)$ and $A^{(-1)} : H^{s-\mu,\gamma-\mu}(M) \to H^{s,\gamma}(M)$ belong to the edge calculus the compositions $A^{(-1)}A, AA^{(-1)}$ are well-defined, and we have $A^{(-1)}A - 1 \in L^0(M,\mathbf{g}_l) \cap L^{-\infty}(s_0(M))$, and $AA^{(-1)} - 1 \in L^0(M,\mathbf{g}_r) \cap L^{-\infty}(s_0(M))$. In order to complete the proof it suffices to apply Remark 2.10. $\square$

Remark 2.22. Observe that the smoothing Mellin plus Green remainders in the relations (2.60) are flat in the sense that locally the occurring smoothing Mellin symbols belong to $C^\infty(\Omega, M_{\mathcal{O}}^{-\infty}(X))$ and the Green symbols to $R_G^0(\Omega \times \mathbb{R}^q)_{\mathcal{O}}$, cf. the notation in Definition 2.4.

Remark 2.23. Let A be elliptic with respect to $\sigma_0(A)$, cf. Definition 2.13. Then $\sigma_c(A)(y,z) := h(0,y,z,0)$ is a holomorphic family of elliptic operators in $L^\mu_{cl}(X)$, smoothly depending on y. For every y there exists a countable set $D(y) \subset \mathbb{C}$ with $D(y) \cap \{c \le \operatorname{Re} z \le c'\}$ finite for every $c \le c'$, such that

(2.61) $$\sigma_c(A)(y,z) : H^s(X) \to H^{s-\mu}(X)$$

is bijective for all $z \in \mathbb{C} \setminus D(y)$ and all s. There is a Mellin asymptotic type $R(y)$ such that $\sigma_c(A)(y,z)^{-1}$ extends from $\mathbb{C} \setminus D(y)$ to an element of $M^{-\mu}_{R(y)}(X)$.

Proposition 2.24. *Let A be elliptic with respect to $\sigma_0(A)$, cf. Definition 2.13, assume that the asymptotic type $R(y)$ of Remark 2.23 does not depend on y, and let (2.61) be bijective for all $z \in \Gamma_{(n+1)/2-\gamma}$ and all y (i.e. $\pi_{\mathbb{C}} R \cap \Gamma_{(n+1)/2-\gamma} = \emptyset$). Then there is an operator $B \in L^{-\mu}_{M+G}(M, \mathbf{g}^{-1})$ such that for $Q := A^{(-1)} + B$ we have*

(2.62)
$$QA = 1 \bmod L^0_G(M, \mathbf{g}_l) + L^{-1}_{M+G}(M, \mathbf{g}_l), \ AQ = 1 \bmod L^0_G(M, \mathbf{g}_r) + L^{-1}_{M+G}(M, \mathbf{g}_r)$$

for $\mathbf{g}_l = (\gamma, \gamma, \Theta), \mathbf{g}_r = (\gamma - \mu, \gamma - \mu, \Theta)$.

Proof. The bijectivity of $\sigma_1(A)(y,\eta)$ as a family of operators (2.39) is an ellipticity condition of the respective cone operators over $X^\wedge$ at $r = 0$ which contains the bijectivity of the principal conormal symbol

(2.63) $$\sigma_c(A)(y,z) = \tilde{h}(0,y,z,0) = h(0,y,z,0) : H^s(X) \to H^{s-\mu}(X)$$

for all $z \in \Gamma_{(n+1)/2-\gamma}$ and all y. From Theorem 2.18 and Lemma 2.21 we conclude $\sigma_1(A^{(-1)}A)(y,\eta) = \sigma_1(A^{(-1)})(y,\eta)\sigma_1(A)(y,\eta)$ and $T^\mu(\sigma_c(A^{(-1)})(y,z))\sigma_c(A)(y,z) = 1 + l(y,z)$ for some $l(y,z) \in C^\infty(\Omega, M_{\mathcal{O}}^{-\infty}(X))$. In the notation of (2.58) (for omitted j) we have $\sigma_c(A^{(-1)})(y,z) = h^{(-1)}(0,y,z,0)$. This gives us the relation

(2.64) $$h^{(-1)}(0,y,z+\mu,0)h(0,y,z,0) = 1 + l(y,z) \ \text{ for all } \ z \in \mathbb{C}.$$

For analogous reasons we have

(2.65) $$h(0,y,z-\mu,0)h^{(-1)}(0,y,z,0) = 1 + m(y,z) \ \text{ for all } \ z \in \mathbb{C}$$

for some $m(y,z) \in C^\infty(\Omega, M_{\mathcal{O}}^{-\infty}(X))$. For the moment we now fix y and show that there is a $g(y,z) \in M_R^{-\infty}(X)$ for some Mellin asymptotic type R such that

(2.66) $$(h^{(-1)}(0,y,z+\mu,0) + g(y,z+\mu))h(0,y,z,0) = 1 \ \text{ for all } \ z \in \mathbb{C}.$$

To this end we write for abbreviation $h^{(-1)}_\mu := h^{(-1)}(0,y,z+\mu,0), g_\mu := g(y,z+\mu), h := h(0,y,z,0)$. Then the ansatz $(h^{(-1)}_\mu + g_\mu)h = 1$ to determine g_μ yields $(h^{(-1)}_\mu + g_\mu)hh^{(-1)}_\mu = h^{(-1)}_\mu$. Using (2.65) it follows that $(h^{(-1)}_\mu + g_\mu)(1 + m_\mu) = h^{(-1)}_\mu$ for $m_\mu := m(y,z+\mu)$. Thus, by virtue of Theorems 1.5 and 1.6, $g_\mu := h^{(-1)}_\mu(1 + m_\mu)^{-1} - h^{(-1)}_\mu$ is as desired. In other words we have

$$\sigma_c(A)(y,z)^{-1} = h^{(-1)}(0,y,z+\mu,0) + g(y,z+\mu),$$

and by assumption we can choose the decomposition on the right of the latter equation in such a way that $g(y, z) \in C^\infty(\Omega, M_R^{-\infty}(X))$ for some y-independent Mellin asymptotic type R. To complete the proof it suffices to set

$$B = r^\mu \mathrm{Op}_y \omega_\eta \mathrm{op}_M^{\gamma - \mu - n/2}(g)(y) \omega'_\eta.$$

In fact, the first relation of (2.62) holds when $\sigma_c(QA) = T^\mu(\sigma_c(Q))\sigma_c(A) = 1$. However, this is true by construction, since $\sigma_c(Q) = h^{(-1)}(0, y, z, 0) + g(y, z)$ and $\sigma_c(A) = h(0, y, z, 0)$, cf. the relation (2.66). The second relation of (2.62) can be verified in an analogous manner. $\qquad\square$

Corollary 2.25. *Let $A \in L^\mu(M, \mathbf{g}), \mathbf{g} = (\gamma, \gamma - \mu, \Theta)$, be as in Proposition 2.24. Then the operator $Q \in L^{-\mu}(M, \mathbf{g}^{-1}), \mathbf{g}^{-1} = (\gamma - \mu, \gamma, \Theta)$, can be found in such a way that*

$$(2.67) \qquad QA = 1 \bmod L_G^0(M, \mathbf{g}_l), \ AQ = 1 \bmod L_G^0(M, \mathbf{g}_r).$$

In fact, if we denote by $M + G \in L_G^0(M, \mathbf{g}_l) + L_{M+G}^{-1}(M, \mathbf{g}_l)$ the remainder in the first relation of (2.62), then we have $(\sum_{j=0}^k (-1)^j (M + G)^j)(1 + M + G) = 1 \bmod L_G^0(M, \mathbf{g}_l)$; here $\Theta = (-(k+1), 0]$. Thus, if first Q is as in (2.62), the composition $Q_l := (\sum_{j=0}^k (-1)^j (M+G)^j)Q$ satisfies the relation $Q_l A = 1 \bmod L_G^0(M, \mathbf{g}_l)$. In a similar manner we find a Q_r such that $AQ_r = 1 \bmod L_G^0(M, \mathbf{g}_r)$. A standard algebraic argument shows that Q_l or, alternatively, Q_r may be taken as the new operator Q to satisfy the relations (2.67).

Remark 2.26. Observe that $g(y, z) \in C^\infty(\Omega, M_R^{-\infty}(X))$ can be directly deduced from $h^{-1}(0, y, z, 0)$ which is by assumption an element of $C^\infty(\Omega, M_R^{-\mu}(X))$, cf. Remark 1.3.

We did not employ so far the condition (ii) of Definition 2.13. This will be the next point. Observe that the ellipticity of A which includes the bijectivity of (2.39) for all $(y, \eta), \eta \neq 0$, entails the bijectivity of (2.61) for all $z \in \Gamma_{(n+1)/2-\gamma}$ and all y. In fact, the bijectivity of

$$\sigma_1(A)(y, \eta) : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\mu, \gamma-\mu}(X^\wedge)$$

means that $\sigma_1(A)(y, \eta)$ as a Fredholm operator between the Kegel-spaces is ell iptic in the cone algebra over $X^\wedge$ for every fixed y and $\eta \neq 0$, and the respective bijectivity of $\sigma_c(A)(y, z)$ belongs to these ellipticity conditions.

Theorem 2.27. *Let $A \in L^\mu(M, \mathbf{g})$ for $\mathbf{g} = (\gamma, \gamma - \mu, \Theta)$ be elliptic in the sense of Definition 2.13, and assume that the asymptotic type $R(y)$ of Remark 2.23 does not depend on y. Then there exists a parametrix $P \in L^{-\mu}(M, \mathbf{g}^{-1})$ for $\mathbf{g}^{-1} = (\gamma - \mu, \gamma, \Theta)$, i.e. $PA = 1 \bmod L^{-\infty}(M, \mathbf{g}_l), AP = 1 \bmod L^{-\infty}(M, \mathbf{g}_r)$ for $\mathbf{g}_l, \mathbf{g}_r$ as in Proposition 2.24.*

Proof. Let us first assume that A is of the special form (2.57) , and let Q be as in Corollary 2.25. The edge symbol $\sigma_1(Q)(y, \eta) : \mathcal{K}^{s-\mu, \gamma-\mu}(X^\wedge) \to \mathcal{K}^{s,\gamma}(X^\wedge)$ is of index 0. In fact, from (2.67) we know that $\sigma_1(Q)(y, \eta)\sigma_1(A)(y, \eta)$ as well as

$\sigma_1(A)(y,\eta)\sigma_1(Q)(y,\eta)$ have the form $1 + \sigma_1(G)(y,\eta)$ for some $G \in L_G^0(M, \mathbf{g}_l)$ (or $L_G^0(M, \mathbf{g}_r)$). Since Green symbols are compact in the Kegel-spaces, it follows that $\sigma_1(Q)(y,\eta)$ takes values in Fredholm operators of index 0. We will construct P by means of the inverse $\sigma_1(P)(y,\eta) := \sigma_1(A)(y,\eta)^{-1}$ which will be found in the form $\sigma_1(P)(y,\eta) = \sigma_1(Q)(y,\eta) + \sigma_1(G_1)(y,\eta)$ for a suitable $G_1 \in L_G^{-\mu}(M, \mathbf{g}^{-1})$. This is all what we need, since we already have $\sigma_0(P) := \sigma_0(Q) = \sigma_0(A)^{-1}$. A final step then allows us to construct P by a formal Neumann series argument, here in the following version. From $\sigma(P) = (\sigma_0(P), \sigma_1(P))$ we pass to an operator $P_0 \in L^{-\mu}(M, \mathbf{g}_l)$ such that $\sigma(P_0) = \sigma(P)$. Then $P_0 A = 1 + C$ for a $C \in L^{-1}(M, \mathbf{g}_l) \cap L^{-\infty}(s_0(M)) = L_{M+G}^{-1}(M, \mathbf{g}_l)$ since $\sigma(P_0 A - 1) = \sigma(P_0)\sigma(A) - 1 = 0$, cf. the notation (2.51). The asymptotic sum $\sum_{j=0}^{\infty}(-1)^j C^j$ exists in the form $1 + D, D \in L^{-1}(M, \mathbf{g}_l)$ (the assumption of Theorem 2.16 on the sequence $(-1)^j C^j, j \in \mathbb{N}$, is satisfied). A similar construction is possible with compositions in the opposite order. Thus $P := (1 + D)P_0$ is as desired. So it remains to construct the above-mentioned $\sigma_1(G_1)(y,\eta)$. First we carry out the construction for all $\{(y,\eta) \in \overline{U}(y_0) \times S^{q-1} : |\eta - \eta_0| < \varepsilon\}$ for a sufficiently small neighbourhood $U(y_0)$ of a point y_0 and sufficiently small $\varepsilon > 0$. Since

$$(2.68) \qquad \sigma_1(Q)(y,\eta) : \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \to \mathcal{K}^{s,\gamma}(X^\wedge)$$

is of index 0 we can fill up (2.68) to a 2×2 block matrix family of bijective operators

$$(2.69) \qquad f(y,\eta) = (f_{ij}(y,\eta))_{i,j=1,2} : \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \oplus \mathbb{C}^N \to \mathcal{K}^{s,\gamma}(X^\wedge) \oplus \mathbb{C}^N$$

for a suitable N and $f_{11} := \sigma_1(Q)$, smooth in $(y,\eta) \in \{U(y_0) \times S^{q-1} : |\eta - \eta_0| < \varepsilon\}$ where f_{12} takes values in $\mathcal{L}(\mathbb{C}^N, C_0^\infty(X^\wedge))$ and $f_{21} = (f_{21,1}, \ldots, f_{21,N})$ is a vector of maps $f_{21,k} : \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \to \mathbb{C}$ defined by $f_{21,k}u := \int_{X^\wedge} f_{21,k}(r,x)u(r,x)drdx$ for certain (y,η)-dependent functions $f_{21,k}(r,x) \in C_0^\infty(X^\wedge)$, $k = 1, \ldots, N$. The right lower corner f_{22} is simply a smooth function in (y,η) with values in $N \times N$ matrices. Generalities on constructions of that kind may be found in [10, Section 3.3.4]. Here we use, in particular, the compactness of the closure of $V_1 := \{(y,\eta) \in U(y_0) \times S^{q-1} : |\eta - \eta_0| < \varepsilon\}$. Since the invertible $N \times N$ matrices form an open dense subset of $\mathbb{C}^{N^2}$ a small perturbation of $f_{22}(y_0, \eta_0)$ allows us to pass to another $f(y,\eta)$ for which $f_{22}(y_0, \eta_0)$ is invertible. In other words, in the choice of (2.69) we may assume the invertibility of $f_{22}(y_0, \eta_0)$. Then $f_{22}(y,\eta)$ is automatically invertible for all (y,η) in a neighbourhood of (y_0, η_0). Thus by choosing the diameter of V_1 sufficiently small we have the invertibility of the right lower corner for all (y,η) in that set. This allows us to pass to a modified upper left corner, namely,

$$(2.70) \qquad \tilde{f}_{11}(y,\eta) := f_{11}(y,\eta) + g_{11}(y,\eta) : \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \to \mathcal{K}^{s,\gamma}(X^\wedge)$$

for $g_{11}(y,\eta) := -(f_{12}f_{22}^{-1}f_{21})(y,\eta)$ which is invertible in a small neighbourhood of (y_0, η_0). Now let us extend $g_{11}(y,\eta)$ by homogeneity to the set $V := \{(y,\eta) \in U(y_0) \times (\mathbb{R}^q \setminus \{0\}) : (y, \eta/|\eta|) \in V_1\}$ by setting

$$(2.71) \qquad g_{11}(y,\eta) := |\eta|^{-\mu}\kappa_{|\eta|}g_{11}(y,\eta/|\eta|)\kappa_{|\eta|}^{-1}$$

By construction $g_{11}(y, \eta)$ is the homogeneous principal part of order $-\mu$ of a flat Green symbol in the conical set V. Let us now form the composition

$$(2.72)$$
$$(\sigma_1(Q) + g_{11})(y, \eta)\sigma_1(A)(y, \eta) = (1 + \sigma_1(G) + f_0)(y, \eta) : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s,\gamma}(X^\wedge)$$

for $f_0 := g_{11}\sigma_1(A), (y, \eta) \in V$ (recall that $\sigma_1(Q) = f_{11}$). The operators (2.72) are invertible for all $(y, \eta) \in V$, and (2.72) has the form $1 + g_0(y, \eta)$ for the homogeneous principal part $g_0(y, \eta) = (\sigma_1(G) + f_0)(y, \eta)$ of order 0 of a corresponding Green symbol. Since f_0 maps to flat functions, the asymptotics of the functions in the image of g_0 and $\sigma_1(G)$ coincide. We have $(1 + g_0(y, \eta))^{-1} = 1 + g_1(y, \eta)$ for the homogeneous principal part g_1 of order 0 of another Green symbol with the same asymptotics in the image as for $\sigma_1(G)$. In fact, $(1 + g_0)(1 + g_1) = 1$ gives us $g_1 = -g_0(1 + g_1)$. We obtain

$$(2.73) \qquad (1 + g_1(y, \eta))(\sigma_1(Q) + g_{11})(y, \eta)\sigma_1(A)(y, \eta) = 1,$$

i.e. $(1 + g_1(y, \eta))(\sigma_1(Q) + g_{11})(y, \eta)$ is a left inverse of $\sigma_1(A)(y, \eta)$ in the set V. In a similar manner we can construct a right inverse, and hence

$$(2.74) \qquad \sigma_1(A)^{-1}(y, \eta) = (1 + g_1(y, \eta))(\sigma_1(Q) + g_{11})(y, \eta).$$

This is true first in V; however, the above-mentioned point (y_0, η_0) is arbitrary. Therefore, we computed the inverse of $\sigma_1(A)(y, \eta)$ everywhere. All compositions are controlled within the algebra of edge symbols. In our case we have $\sigma_1(G_1)(y, \eta) :=$ $g_1(y, \eta)\sigma_1(Q)(y, \eta) + g_1(y, \eta)g_{11}(y, \eta) + g_{11}(y, \eta)$ which is of Green type.

Let us now assume that $A \in L^\mu(M, \mathbf{g})$ also contains a non-trivial Mellin plus Green summand, i.e. A can be written as

$$(2.75) \qquad\qquad A = A_0 + M + G + C$$

where A_0 is of the form (2.57). Clearly we may assume $C = 0$. We first show an analogue of Proposition 2.24; then the rest of the proof is exactly as before. In the present case instead of (2.63) we have

$$(2.76) \qquad\qquad \sigma_c(A)(y, z) = h(0, y, z, 0) + \sigma_c(M)(y, z)$$

where $\sigma_c(M)(y, z) = f_{00}(y, z)$ when M is given in the form (2.26). It suffices to show the existence of a $g(y, z) \in C^\infty(\Omega, M_R^{-\infty}(X))$ such that

$$(h^{-1}(0, y, z + \mu, 0) + g(y, z + \mu))(h(0, y, z, 0) + f_{00}(y, z)) = 1.$$

Let us write again $h_\mu := h^{(-1)}(0, y, z + \mu, 0), g_\mu := g(0, z + \mu), h := h(0, y, z, 0)$ and make the ansatz $(h_\mu^{(-1)} + g_\mu)(h + f_{00}) = 1$ to determine g_μ. Applying Theorem 1.7 we see that (y-wise) $(h + f_{00})^{-1} = h_\mu^{(-1)} + r$ for some $r \in M_R^{-\infty}(X)$. This gives us $h_\mu^{(-1)} + g_\mu = h_\mu^{(-1)} + r$ and hence $r = g_\mu$. Finally if $A^{(-1)}$ is associated with A_0 as in Proposition 2.24 we find a $B \in L_{M+G}^{-\mu}(M, \mathbf{g}^{-1})$ such that $Q := A^{(-1)} + B$ has the properties (2.62), first from the left, and then, analogously, from the right. It follow a refined Q as in Corollary 2.25 such that the relations (2.67) hold. Then we proceed as in the first part of the proof and obtain a two-sided parametrix P of (2.75). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 2.28. With $P \in L^{-\mu}(M, \mathbf{g}^{-1})$ also $P + C$ for arbitrary $C \in L^{-\infty}(M, \mathbf{g}^{-1})$ is a parametrix of $A \in L^{\mu}(M, \mathbf{g})$; moreover, if $P, Q \in L^{-\mu}(M, \mathbf{g}^{-1})$ are parametrices of A, then $P - Q \in L^{-\infty}(M, \mathbf{g}^{-1})$.

Theorem 2.29. *Let* $A \in L^{\mu}(M, \mathbf{g})$ *be as in Theorem 2.27. Then* $Au = f$ *for* $u \in H^{-\infty,\gamma}(M)$ *and* $f \in H^{s-\mu,\gamma-\mu}(M), s \in \mathbb{R}$, *entails* $u \in H^{s,\gamma}(M)$. *If in addition* $f \in H_S^{s-\mu,\gamma-\mu}(M)$ *for an asymptotic type* S *associated with* $(\gamma - \mu, \Theta)$ *it follows that* $u \in H_R^{s,\gamma}(M)$ *for a resulting asymptotic type* R *associated with* (γ, Θ).

Proof. Let $P \in L^{-\mu}(M, \mathbf{g}^{-1})$ a parametrix of A. Then $Au = f$ implies $PAu = (1 + C)u = Pf$, i.e. $u = Pf - Cu$, and it suffices to apply (2.52). $\qquad \square$

Remark 2.30. The resulting asymptotic type in Theorem 2.29 is not affected by the choice of the parametrix P. In fact, if $P + D, D \in L^{-\infty}(M, \mathbf{g}^{-1})$, is another parametrix of A, cf. Remark 2.28, then $(P + D)Au = (1 + C)u + DAu = (P + D)f$ gives us $u = (P + D)f - DAu - Cu = Pf - Cu$.

Let us now turn to asymptotic parametrices. The general idea is similar as in the conical case. Let $A \in L^{\mu}(M, \mathbf{g})$ be as in Theorem 2.27, and let $P \in L^{-\mu}(M, \mathbf{g}^{-1})$ be a parameterix. Then applying Proposition 2.11 to P we obtain an expansion

$$(2.77) \qquad P = \sum_{m=0}^{N} \mathbf{r}^m P_m \mod L_G^{-\mu}(M, \mathbf{g}^{-1})$$

for every $N \in \mathbb{N}$. Let us call an expansion (2.77) an asymptotic parametrix of A. Writing A itself in the form (2.36) the operator A_0 is elliptic in $L^{\mu}(M, \mathbf{g})$, and $P_0 \in L^{-\mu}(M, \mathbf{g}^{-1})$ in (2.77) can be taken as the parametrix of A_0, obtained by Theorem 2.27. The construction of P_0 is easier than that of P itself because P_0 concerns the case of constant coefficients with respect to r. Now the idea of asymptotic parametrices is to obtain (2.77) by an iterative process, starting with the ansatz

$$(2.78) \qquad \Big(\sum_{m=0}^{N} \mathbf{r}^m P_m \Big) \Big(\sum_{i=0}^{N} \mathbf{r}^i A_i \Big) \sim 1, \ \Big(\sum_{i=0}^{N} \mathbf{r}^i A_i \Big) \Big(\sum_{m=0}^{N} \mathbf{r}^m P_m \Big) \sim 1;$$

$\sim$ means equality modulo a Green plus flat remainder (the remainders will be characterised below). Let us discuss the first relation of (2.78); the second one is similar. The operator P_0 is already the parametrix of A_0. In the construction we first replace $\mathbf{r}^m P_m$ by $\boldsymbol{P_m}$, where $\boldsymbol{P_0} := P_0$. For $\boldsymbol{P_1}$ we set

$$(2.79) \qquad \boldsymbol{P_1} := -P_0 \mathbf{r} A_1 P_0.$$

This operator satisfies the relation

$$(2.80) \qquad \boldsymbol{P_1} A_0 + P_0 \mathbf{r} A_1 = -P_0 \mathbf{r} A_1 C_l \in L^{-\infty}(M, \mathbf{g}_l)$$

where $C_l := P_0 A_0 - 1 \in L^{-\infty}(M, \mathbf{g}_l)$. In general, for $j > 0$, motivated by the desirable identity $\sum_{m+i=j} \boldsymbol{P_m} \mathbf{r}^i A_i = 0$, or $\boldsymbol{P_j} A_0 = -\sum_{m+i=j, m<j} \mathbf{r}^m P_m \mathbf{r}^i A_i$,

we set

$$(2.81) \qquad \boldsymbol{P_j} := -\Big(\sum_{m+i=j, m<j} \boldsymbol{P_m} \mathbf{r}^i A_i \Big) P_0,$$

and obtain

$$(2.82) \qquad \sum_{m+i=j} \boldsymbol{P_m} \mathbf{r}^i A_i = -\Big(\sum_{m+i=j, m<j} \boldsymbol{P_m} \mathbf{r}^i A_i \Big) C_l \in L^{-\infty}(M, \mathbf{g}_l)$$

for any j. Also in the edge case the operators (2.81) do not automatically coincide with those in (2.77).The operators (2.81) can be expressed in terms of P_0 and $\mathbf{r}^i A_i$, for $i = 1, \ldots, j$, alone. For instance, we have anlogues of the formulas (1.46), (1.47), etc. Also the other elements of the asymptotic parametrix construction of Section 1.3 can be carried out for the edge case. In particular we obtain the following result.

Theorem 2.31. *Let $A \in L^\mu(M, \mathbf{g})$ be an elliptic operator. Then for any fixed $N \in \mathbb{N}, N \geq 1$, the asymptotic parametrix P_{as} can be chosen in such a way that*

$$(2.83) \qquad P_{\mathrm{as}} A = 1 + C + F$$

for an $F \in L^\mu(M, \mathbf{g}_l)$ which is flat of order $N - 1$ and a $C \in L^\infty(M, \mathbf{g}_l)$.

Remark 2.32. We have a staightforward analogue of Corollary 1.20 in the case of a manifold M with edge.

Some elements of the parametrix construction can be organised in an alternative way when A is an edge-degenerate differential operator. In this case we have locally near the edge $A = r^{-\mu} \mathrm{Op}_y \mathrm{op}_M^{\gamma - n/2}(h)$ for a Mellin symbol $h(r, y, z, \eta) = \sum_{j+|\alpha| \leq \mu} a_{j\alpha}(r, y) z^j (r\eta)^\alpha, a_{j\alpha}(r, y) \in C^\infty(\overline{\mathbb{R}}_+, \mathrm{Diff}^{\mu-(j+|\alpha|)}(X))$, cf. the formula (0.7). Let us consider

$$(2.84) \qquad A_0 = r^{-\mu} \mathrm{Op}_y \mathrm{op}_M^{\gamma - n/2}(h_0) \text{ for } h_0(r, y, z, \eta) = \sum_{j+|\alpha| \leq \mu} a_{j\alpha}(0, y) z^j (r\eta)^\alpha,$$

and construct P_0 under the assumption of the ellipticity of A_0. The other summands of (2.77) can be obtained by the above-mentioned iterative process. We employ the identity

$$(2.85) \qquad r^{-\mu} \mathrm{op}_M^{\gamma - n/2}(h_0) r^\mu = \mathrm{op}_M^{\gamma - n/2 - \mu}(T^{-\mu} h_0)$$

and compute a Mellin symbol $m^{(-1)}$ by the ansatz

$$(2.86) \qquad \begin{aligned} & \big(\mathrm{Op}_y r^{-\mu} \mathrm{op}_M^{\gamma - n/2}(h_0) \big) \big(\mathrm{Op}_y r^\mu \mathrm{op}_M^{\gamma - n/2 - \mu}(m^{(-1)}) \big) \\ & = \big(\mathrm{Op}_y \mathrm{op}_M^{\gamma - n/2 - \mu}(m) \big) \big(\mathrm{Op}_y \mathrm{op}_M^{\gamma - n/2 - \mu}(m^{(-1)}) \big) \sim 1. \end{aligned}$$

for $m(r, y, z, \eta) := \tilde{m}(y, z, r\eta), \tilde{m}(y, z, \tilde{\eta}) := (T^{-\mu} \tilde{h}_0)(y, z, \tilde{\eta})$ where $h_0(r, y, z, \eta) = \tilde{h}_0(y, z, r\eta)$. The meaning of $\sim$ will be specified below.

The ellipticity of A_0 has the consequence that $\tilde{m}(y, z, \tilde{\eta}) \in C^\infty(\Omega, L_{\mathrm{cl}}^\mu(X; \Gamma_\beta \times \mathbb{R}_{\tilde{\eta}}^q))$ is parameter-dependent elliptic, for every $\beta \in \mathbb{R}$. Choose a parameter-dependent

parametrix $\tilde{f}(y, z, \tilde{\eta}) \in C^\infty(\Omega, L_{\mathrm{cl}}^{-\mu}(X; \Gamma_\beta \times \mathbb{R}_{\tilde{\eta}}^q))$ for fixed β (a simple considera-
tion shows that this can be done including smooth dependence in $y \in \Omega$.) Applying
the kernel cut-off operator to $\tilde{f}(y, z, \tilde{\eta})$ gives us a $\tilde{k}_0(y, z, \tilde{\eta}) \in C_0^\infty(\Omega, M_\mathcal{O}^{-\mu}(X; \mathbb{R}_{\tilde{\eta}}^q))$
such that also $\tilde{k}_0(y, z, \tilde{\eta})$ is a parameter-dependent parametrix for $(z, \tilde{\eta}) \in \Gamma_\beta \times \mathbb{R}^q$.
Then, since $\tilde{m}(y, z, \tilde{\eta})\tilde{k}_0(y, z, \tilde{\eta})$ is equal to 1 for all $(z, \tilde{\eta}) \in \Gamma_\beta \times \mathbb{R}^q$ modulo
$C^\infty(\Omega, L^{-\infty}(X; \Gamma_\beta \times \mathbb{R}_{\tilde{\eta}}^q))$ it follows also a similar eqivalence for all $(z, \tilde{\eta}) \in$
$\Gamma_\delta \times \mathbb{R}^q$ for every real δ. Let us now find a Mellin symbol as an asymptotic sum
$\tilde{m}^{(-1)}(r, y, z, \tilde{\eta}) \sim \sum_{\kappa=0}^\infty \tilde{k}_\kappa(r, y, z, \tilde{\eta})$ in $C^\infty(\overline{\mathbb{R}}_+ \times \Omega, M_\mathcal{O}^{-\mu}(X; \mathbb{R}_{\tilde{\eta}}^q))$ with $k_\kappa(r, y, z,$
$\eta) = \tilde{k}_\kappa(r, y, z, r\eta)$, by successively solving the equations

$$(2.87) \qquad \sum_{|(\alpha_0, \alpha')| + \kappa = l} 1/\alpha_0! \alpha'! \left(\partial_z^{\alpha_0} \partial_\eta^{\alpha'} m\right)\left((-r\partial_r)^{\alpha_0} D_y^{\alpha'} k_\kappa\right) = 0 \text{ for all } l = 1, 2, \ldots.$$

$\tilde{k}_0(y, z, \tilde{\eta})$ is already determined. For $l = 1$ we obtain

$$(2.88) \qquad k_1 = -k_0 \sum_{|\alpha_0, \alpha'| = 1} \left(\partial_z^{\alpha_0} \partial_\eta^{\alpha'} m\right)\left((-r\partial_r)^{\alpha_0} D_y^{\alpha'} k_0\right)$$

which is of the form $k_1(r, y, z, \eta) = \tilde{k}_1(r, y, z, r\eta)$ for a $\tilde{k}_1(r, y, z, \tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times$
$\Omega, M_\mathcal{O}^{-\mu-1}(X; \mathbb{R}_{\tilde{\eta}}^q))$. More generally, the solution of (2.87) for arbitrary $l \geq 1$ has
form $k_l(r, y, z, \eta) = \tilde{k}_l(r, y, z, r\eta)$ for a $\tilde{k}_l(r, y, z, \tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, M_\mathcal{O}^{-\mu-l}(X; \mathbb{R}_{\tilde{\eta}}^q))$.
In other words the above-mentioned asymptotic sum for $\tilde{k}(r, y, z, \tilde{\eta})$ can be carried
out in the desired form. Note that, although in the case of a differential operator
this process is not finite, we always have $\partial_z^{\alpha_0} \partial_\eta^{\alpha'} m = 0$ as soon as $|\alpha_0, \alpha'| > \mu$,
i.e. in (2.87) it suffices to take the sum over all $|\alpha_0, \alpha'| \leq \mu$. Clearly the indicated
method of constructing a parametrix in the edge calculus also works in the pseudo-
differential case. What we did so far is a new computation of the non-smoothing
holomorphic Mellin symbol for the parametrix, here carried out for A_0. As we
see, this Mellin symbol is not necessarily constant in r near 0. However, in the
constructed asymptotic sum for $\tilde{m}^{(-1)}$ we can easily split up the r-independent
contributions. The remaining part of the parametrix construction is similar as
before.

2.4. Tools from the singular analysis

We first recall the definition of the weighted Kegel-spaces $\mathcal{K}^{s,\gamma}(X^\wedge)$ which is most
simple in the case of the unit sphere S^n in $\mathbb{R}^{n+1}$. In Section 1.1 we defined the
weighted spaces $\mathcal{H}^{s,\gamma}(X^\wedge)$. For any fixed cut-off function ω on the half-axis we set

$$\mathcal{K}^{s,\gamma}((S^n)^\wedge) := \{\omega u + (1 - \omega)v : u \in \mathcal{H}^{s,\gamma}((S^n)^\wedge), v \in H^s(\mathbb{R}^{n+1})\}$$

for $s, \gamma \in \mathbb{R}$. The definition is independent of the choice of ω. The definition for gen-
eral X refers to a localisation on $U^\wedge = \mathbb{R}_+ \times U$ for any coordinate neighbourhood
U on X. Let us fix a diffeomorphism

$$\chi : \mathbb{R}_+ \times U \to \Gamma$$

for a conical set $\Gamma \subset \mathbb{R}^{n+1} \setminus \{0\}$ such that $\chi(r,x) = r\chi(1,x), r > 0$, for a diffeomorphism $\chi(1,\cdot) : U \to \Gamma \cap S^n$. Then $\mathcal{K}^{s,\gamma}(X^\wedge)$ is defined to be the set of all $u \in H^s_{\text{loc}}(X^\wedge)$ such that $\varphi u \circ \chi^{-1} \in \mathcal{K}^{s,\gamma}((S^n)^\wedge)$ for every such χ and every $\varphi \in C_0^\infty(U)$.

From the definition it follows that

$$\mathcal{K}^{0,0}(X^\wedge) = \mathcal{H}^{0,0}(X^\wedge) = r^{-n/2} L^2(\mathbb{R}_+ \times X).$$

In $\mathcal{K}^{s,\gamma}(X^\wedge)$ we can easily introduce a scalar product that turns the space to a Hilbert space. In $\mathcal{K}^{s,\gamma}(X^\wedge)$ we consider the group action $\kappa = \{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$ defined as

$$\kappa_\lambda : u(r,x) \to \lambda^{n/2} u(\lambda r, x), \ \lambda \in \mathbb{R}_+,$$

cf. the notation in connection with (2.14). Also the spaces $\mathcal{K}^{s,\gamma;e}(X^\wedge) := \langle r \rangle^{-e} \mathcal{K}^{s,\gamma}(X^\wedge), s, \gamma, e \in \mathbb{R}$, and the Fréchet subspaces with asymptotics $\mathcal{K}_P^{s,\gamma;e}(X^\wedge)$ admit the action of κ.

Let us now recall the definition of operator-valued symbols in the set-up of group actions in the involved spaces. Let H and $\tilde{H}$ be Hilbert spaces with group actions κ and $\tilde{\kappa}$, respectively. Then

(2.89) $S^\mu(U \times \mathbb{R}^q; H, \tilde{H})$ for $\mu \in \mathbb{R}$ and open $U \subseteq \mathbb{R}^p$

is defined to be the set of all $a(y,\eta) \in C^\infty(U \times \mathbb{R}^q, \mathcal{L}(H, \tilde{H}))$ satisfying the symbolic estimates

(2.90) $\|\tilde{\kappa}_{\langle \eta \rangle}^{-1} \{D_y^\alpha D_\eta^\beta a(y,\eta)\} \kappa_{\langle \eta \rangle}\|_{\mathcal{L}(H, \tilde{H})} \leq c \langle \eta \rangle^{\mu - |\beta|}$

for all $(y,\eta) \in K \times \mathbb{R}^q$ for any $K \subset\subset U$ and all multi-indices $\alpha \in \mathbb{N}^p, \beta \in \mathbb{N}^q$, for constants $c = c(\alpha, \beta, K) > 0$. The elements of (2.89) are referred to as operator-valued symbols (in the frame of twisted homogeneity). By twisted homogeneity of order μ of a function $a_{(\mu)}(y,\eta) \in C^\infty(U \times (\mathbb{R}^q \setminus \{0\}), \mathcal{L}(H, \tilde{H}))$ we understand the relation

(2.91) $a_{(\mu)}(y, \lambda \eta) = \lambda^\mu \tilde{\kappa}_\lambda a_{(\mu)}(y,\eta) \kappa_\lambda^{-1}$ for all $\lambda \in \mathbb{R}_+$.

An $a_\mu(y,\eta) \in C^\infty(U \times \mathbb{R}^q, \mathcal{L}(H, \tilde{H}))$ is said to be twisted homogeneous of order μ for large $|\eta|$ if there is a constant $C > 0$ such that the relation $a_\mu(y, \lambda \eta) = \lambda^\mu \tilde{\kappa}_\lambda a_\mu(y,\eta) \kappa_\lambda^{-1}$ holds for all $|\eta| \geq C$ and $\lambda \geq 1$. We often employed the fact that any such $a_\mu(y,\eta)$ belongs to the space (2.89). By

(2.92) $S_{\text{cl}}^\mu(U \times \mathbb{R}^q; H, \tilde{H})$

we denote the subset of all (classical) elements $a(y,\eta)$ of (2.89) such that there are functions $a_{\mu-j}(y,\eta) \in C^\infty(U \times \mathbb{R}^q, \mathcal{L}(H, \tilde{H}))$, twisted homogeneous of order $\mu - j$ for large $|\eta|$, such that $a(y,\eta) - \sum_{j=0}^N a_{\mu-j}(y,\eta) \in S^{\mu-(N+1)}(U \times \mathbb{R}^q; H, \tilde{H})$ for every $N \in \mathbb{N}$.

These notions have a straighforward generalisation to the case of Fréchet spaces E and/or $\tilde{E}$ with group action κ and $\tilde{\kappa}$, respectively. In particular, this is applied in Definition 2.4.

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